Fuzzy belief structures viewed as classical belief structures: a practical viewpoint.

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***Abstract*—Many authors have studied fuzzy belief structures, that is belief functions having fuzzy sets as focal elements. One of the main reason for this is that this structure offers a convenient way to mix probabilistic and fuzzy information. Still, one point on which authors often disagree is how information represented by fuzzy belief structures should be processed, i.e., how should be defined fusion operations, decision rules, uncertainty measures, uncertainty propagation, etc. for such representations. In this paper, we consider that fuzzy belief structures are mapped into classical belief structures encoding the same information, and propose to manipulate these latter structures. From a practical standpoint, it has the benefit that processing tools proper to belief structures can be used together with their interpretation, rather than having to consider a mix of probabilistic and fuzzy calculi, which can be harder to interpret.**

I. Introduction

The uncertainty concerning the value of a variable can come in different forms (vague assessments, set of measurements, imprecise observations, unreliability, ...), requiring different representations. Among existing models, probability distributions and fuzzy sets are the simplest representations to respectively model imprecision and randomness in the observations. As both type of uncertainty often coexist in practical applications, it is desirable to integrate fuzzy sets and probability distributions in a single representation and to treat both uncertainty types at once.

Fuzzy belief structures [1] present a natural answer to such a requirement. Such structures are defined by probability masses summing up to one and bearing no longer on single points but on fuzzy sets. Formally, they are equivalent to so-called fuzzy random variables or random fuzzy sets [2]. Fuzzy belief structures can be the result of different information processing (information fusion [3], propagation [4], linguistic probability assessments [5], ...), can address a variety of problems (control [6], optimisation under uncertainty [7]) and be considered with different semantics (fuzzy sets can either be considered as elements of a particular domain [8] or as the imprecise description of our knowledge about a variable [2]). Due to this variety of origins and purposes, many different ways in which information represented by a fuzzy belief structure should be processed have been proposed, making the interpretation and practical handling of fuzzy belief structures difficult for non-expert users.

Some results [9] show that a fuzzy belief structure can be mapped into a classical belief structure capturing the same information as the original fuzzy belief structure. This view allows to use processing tools proper to belief structures, without having to mix them with fuzzy calculus. In this paper, we investigate the practical handling of fuzzy belief structures under such a view, and links it to propositions separately handling fuzzy and probabilistic information when possible.

After some preliminaries (Section II), we investigate more specifically the following problems: information fusion (Section III), focusing on the interpretation of Dempster's rule; information propagation (Section IV), in particular the problems of independence modelling and of cascading propagation; decision rules (Section V).

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II. Fuzzy belief structures and random sets

Note that, in this paper, fuzzy belief structures represent our uncertainty about the true (crisp) value of a variable X assuming its value on a domain $\mathcal{X}$ that either is finite or is some (discretized) product space of the real line. Fuzzy sets are thus here interpreted as possibility distributions, and we do not consider other semantics [10]. Therefore, we will indifferently refer to a fuzzy set or to the possibility distribution equivalent to its membership function.

A. Possibility distributions

A possibility distribution $\pi(x)$ is a mapping from a space $\mathcal{X}$ to $[0,1]$ and is formally equivalent to a fuzzy membership function μ s.t. $\mu(x)=\pi(x)$. One can interpret a possibility distribution on the real line as a set of nested confidence intervals [11]. From a possibility distribution, several set-functions can be defined [12]: the possibility, necessity and sufficiency measures, respectively defined as

$$\Pi(A)=\sup_{x\in A}\pi(x)$$

$$N(A)=1-\Pi(A^c)$$

$$\Delta(A)=\inf_{x\in A}\pi(x)$$

Where A^c stands for the complement of A. A possibility degree $\Pi(A)$ quantifies to what extent the event A is plausible, the necessity degree quantifies the certainty of A and the sufficiency degree quantify the guaranteed possibility of an event A. These measures can sometimes be interpreted as probability bounds, in the sense that a distribution π induces a probability set $\mathcal{P}_\pi$ containing the probability measures dominating the necessity measure, i.e.

$$\mathcal{P}_\pi=\{p\in\mathbb{P}_\mathcal{X}|\forall A\subseteq\mathcal{X},P(A)\geq N(A)\},$$

with P the probability measure induced by the probability mass function p, and $\mathbb{P}_{\mathcal{X}}$ the set of probability mass functions over $\mathcal{X}$

An α-cut E_α of the distribution π is defined as the set

$$E_\alpha = \{x | \pi(x) \geq \alpha\}$$

The core $c(\pi)$ and the support $s(\pi)$ of π respectively correspond to E_1 and $\lim_{\epsilon \to 0} E_\epsilon$

B. Belief structure (BF)

A belief structure consists of a mapping m from subsets of a space $\mathcal{X}$ to $[0,1]$ s.t. $\sum_{E \subseteq X} m(E) = 1$,$m(E) \geq 0$ and $m(\emptyset) = 0$. Sets E that have positive mass are called focal sets. From this mapping, we can again define three set-functions, the plausibility, belief and commonality functions, which read [13]:

$$Bel(A) = \sum_{E, E \subseteq A} m(E)$$

$$Pl(A) = \sum_{E, E \cap A} m(E) = 1 - Bel(A^c)$$

$$Q(A) = \sum_{E, E \supseteq A} m(E)$$

where the belief function quantifies the amount of information that surely supports A, the plausibility reflects the amount of information that potentially supports A and the commonality function the amount of information implied by A. In this model the mass $m(E)$ should be interpreted as the probability of only knowing that the unknown quantity lies in E. When focal sets are nested, a belief structure is equivalent to a possibility distribution, and the belief (resp. plausibility and commonality) function is also a necessity (resp. possibility and sufficiency) measure. Such belief structures are usually called *consonant*. Let F be a fuzzy set and $\alpha_0 = 0 < \alpha_1 < \ldots < \alpha_N = 1$ the set of its distinct membership function values. Then F induce (and is induced) by the belief structure having, for $j = 1, \ldots, N$, the focal elements F_j such that

$$m(F_j) = \alpha_j - \alpha_{j-1},$$

with F_j the α_j-cut of F.

Note that a belief structure m also induces a particular probability set $\mathcal{P}_m$ such that

$$\mathcal{P}_m = \{p \in \mathbb{P}_{\mathcal{X}} | \forall A \subseteq \mathcal{X}, P(A) \geq Bel(A)\}.$$

This interpretation of belief structures inherits from Dempster's [14] work.

C. Fuzzy belief structure (FBS)

Zadeh [15] was the first to propose an extension of belief structures when focal sets are fuzzy sets. Here, we will note these fuzzy sets F_i, with $\alpha_0 = 0 < \alpha_1 < \ldots < \alpha_N = 1$ the distinct values of the membership function of every F_i. Since then, many proposals have appeared that extend plausibility and belief functions when focal elements also are fuzzy (see, for example [16], [1]). In this paper, we retain Yen's [17] definition, which, in the discrete case, reads:

$$Pl_m(A) = \sum_{i=1}^{n} m(F_i) \sum_{j=1}^{N} (\alpha_j - \alpha_{j-1}) \max_{w \in F_{i,j}} \mu_A(w) \quad (1)$$

$$Bel_m(A) = \sum_{i=1}^{n} m(F_i) \sum_{j=1}^{N} (\alpha_j - \alpha_{j-1}) \min_{w \in F_{i,j}} \mu_A(w) \quad (2)$$

where $F_{i,j}$ is the α_j-cut of the fuzzy focal element F_i. The reason of choosing this generalization rather than another one is that the part involving fuzzy sets F_i in equations (1) and (2) is equivalent to computing the Choquet integral [18] of the (possibly fuzzy) event A over the possibility distribution $\pi_i = \mu_{F_i}$. We thus use linear operators in every part of the equation, which seems to us more coherent than using a mixing of linear operators and maximum/minimum based operators. Let us also notice that Yen's approach is a generalization of Smet's definition [19]. Although Yen's work is not based on these two considerations, but rather on optimization criteria, it is interesting to underly the fact that this generalization of belief structures to fuzzy focal sets has strong theoretical justifications and interconnections with other theories. The commonality measure can be extended likewise and be defined for fuzzy belief structures and fuzzy events. If A is such a fuzzy event, then the commonality measure $Q_m(A)$ induced by a fuzzy belief structure can be defined as

$$Q_m(A) = \sum_{i=1}^{n} m(F_i) \sum_{j=1}^{N} (\alpha_j - \alpha_{j-1}) \inf_{w \in (F_{i,j})^c} (1 - \mu_A(w)), \quad (3)$$

with E^c the complement of E and the convention that $\inf_{w \in E^c}(1 - \mu_A(w)) = 1$ if $E = \mathcal{X}$. It can easily be checked that if A is a crisp event and if m is reduced to a classical belief structure or to a single fuzzy set, we respectively retrieve the commonality measure and the sufficiency measure.

In fact, viewing the fuzzy belief structure as presented here comes down to reducing a random fuzzy set to a random set where each cut $F_i^{\alpha_j}$ has mass $m(F_i)(\alpha_j - \alpha_{j-1})$ [4]. In the continuous case [20], it is equivalent to consider the convex combination of possibility and necessity measures (viewed as continuous consonant plausibility and belief functions) induced by π_i.

In order to differentiate between a fuzzy belief structure (i.e. focal weights affected to fuzzy sets) and its mapping into a classical belief structure (i.e. weights given to sets that are α-cuts of fuzzy focal elements) when needed, we will use the notation $\tilde{m}$ for the former and m for the latter. We will also speak of belief structures when referring to classical belief structures. Also, when a clear separation between processing operations will be needed, we will refer to the probabilistic information of the fuzzy belief structure when processing operations concern the mass assignments of the fuzzy focal elements, and to the fuzzy information of the fuzzy belief structure when processing operations concern the fuzzy focal sets.

Also note that for any event $A \subseteq \mathcal{X}$, we have $Pl_m(A) = \sum_{i=1}^{n} m(F_i)\Pi_i(A)$, with $\Pi_i(A)$ the possibility measure induced by F_i. This means that we can also interpret this mapping of a fuzzy belief structure into a classical belief structure in terms of induced probability sets, since we have that $\mathcal{P}_m$, the probability set induced by Pl_m, is the convex combination of each $\mathcal{P}_{F_i}$, i.e.

$$\mathcal{P}_m = \left\{ \sum_{i=1}^{n} m(F_i)p_i \in \mathbb{P}_{\mathcal{X}} | p_i \in \mathcal{P}_{F_i} \right\}, \tag{4}$$

with $\mathcal{P}_{F_i}$ the probability set induced by F_i. Finally, it is useful to note that if any fuzzy belief structure can be mapped into a belief structure, the reverse is also true, that is any belief structure can be mapped into a (non-unique) fuzzy belief structure by rearranging focal elements into nested subgroups, and affecting the sum of these focal elements weight as the weight of the resulting fuzzy focal element. The mapped fuzzy focal elements may be restricted to fuzzy subsets taking their membership values in $\{0, 1\}$, retrieving crisp focal sets in this case. We can now study the handling of fuzzy belief structures under this particular assumption, seeing them as belief structures. We first introduce an example that we will use in the sequel to illustrate our purpose.

Example 1. We consider as domain $\mathcal{X}$ the interval $[1, 9]$ discretized into three elements $x_1 = [1, 3]; x_2 =]3, 6]; x_3 =]3, 9]$.

A first fuzzy belief structure $\tilde{m}_1$ is defined on two fuzzy focal elements F_1 and F_2, with $\tilde{m}_1(F_1) = 0.8$, $\tilde{m}_1(F_2) = 0.2$ and F_1, F_2 summarized in the following table

	x_1	x_2	x_3
F_1	1	1	0.5
F_2	0	1	1

This fuzzy belief structure can be mapped into the belief structure m_1 such that

$$m_1(\{x_1, x_2\}) = 0.4 \quad m_1(\mathcal{X}) = 0.4$$
$$m_1(\{x_2, x_3\}) = 0.2$$

A second fuzzy belief structure $\tilde{m}_2$, reduced to the single fuzzy set G_1, is summarized in the following table

	x_1	x_2	x_3
G_1	0.2	1	1

and is equivalent to the belief structure m_2 such that

$$m_2(\{x_2, x_3\}) = 0.8 \quad m_2(\mathcal{X}) = 0.2$$

III. Fusion

Let us first consider the problem of combining a set of fuzzy belief structures provided by different sources. Several authors have proposed fusion rules to merge a set of fuzzy belief structures, all of them combining Dempster's rule of combination with fuzzy combination rules (i.e., t-norms [21] and t-conorms), respectively to combine probabilistic and fuzzy information of the fuzzy belief structures. One of the reason for this is that they interpret fuzzy focal elements as fuzzy propositions (which is not the case here).

However, there are situations where such combinations could be questioned, for instance when the fuzzy sets describe the vague knowledge one has about a precise value, that is when they describe our uncertainty about the value of a variable (e.g., fuzzy sets describing the age of an individual). Also, it can be difficult to give some interpretation in term of source or evidence dependencies when different combination operators are used within the same representation.

Let $\tilde{m}_1$ and $\tilde{m}_2$ be two fuzzy belief structures, with respectively $F_1, \ldots, F_L$ and $G_1, \ldots, G_K$ their focal elements. When one considers their mappings into belief structures m_1, m_2, fusion rules coming from evidence theory [22] and the like can be applied straightforwardly. In particular, if information sources are assumed to be independent, we can simply apply Dempster's rule, giving to every subset $A \subseteq \mathcal{X}$ the mass $m_{1\otimes 2}$ such that

$$m_{1\otimes 2}(A) = \sum_{\substack{F_{i,j} \cap G_{k,l} = A \\ i \in [1,L], j \in [1,N_1] \\ k \in [1,K], l \in [1,N_2]}} m(F_i)m(G_k)(\alpha_j - \alpha_{j-1})(\beta_l - \beta_{l-1}),$$

with $\alpha_0 = 0 < \alpha_1 < \ldots < \alpha_{N_1} = 1$ (resp. $\beta_0 = 0 < \beta_1 < \ldots < \beta_{N_2} = 1$) the set of distinct membership function values of fuzzy focal elements F_i (resp. G_k), and $F_{i,j}$ (resp. $G_{k,l}$) is the α_j-cut of the fuzzy set F_i (resp. the β_l-cut of G_k).

In the case of consonant belief structures (i.e., single fuzzy sets), Dubois and Prade [23] have shown that the upper measures obtained by taking the product combination of the fuzzy sets and by using Dempster's rule (without normalisation) coincide on singletons. This shows that the result obtained with the product t-norm can be seen as a partial information of the result obtained by Dempster's rule, and can therefore be assimilated to an assumption of independence. The next proposition shows that we have a similar property here.

Proposition 1. *Let $\tilde{m}_1, \tilde{m}_2$ be two fuzzy belief structures, then the following equality holds for any $x \in \mathcal{X}$*

$$Pl_{1\otimes 2}(\{x\}) = \sum_{\substack{F_i, G_k \\ i \in [1,L], k \in [1,K]}} m(F_i)m(G_k)\mu_{F_i}(x)\mu_{G_k}(x), \tag{5}$$

with $Pl_{1\otimes 2}$ the plausibility measure induced by $m_{1\otimes 2}$ and μ_F the membership function value of F.

Proof: First, consider the equation of $Pl_{1\otimes 2}(\{x\})$, i.e.,

$$Pl_{1\otimes 2}(\{x\}) = \sum_{\substack{x \in F_{i,j} \cap G_{k,l} \\ i \in [1,L], j \in [1,N_1] \\ k \in [1,K], l \in [1,N_2]}} m(F_i)m(G_k)(\alpha_j - \alpha_{j-1})(\beta_l - \beta_{l-1}) \tag{6}$$

Now, consider the product $\mu_{F_i}(x)\mu_{G_k}(x)$ of the two membership functions. Considering the equivalent consonant belief structures and using Dubois and Prade result [23], we have

$$\mu_{F_i}(x)\mu_{G_k}(x) = \sum_{\substack{x\in F_{i,j}\cap G_{k,l}\\ j\in[1,N_1],l\in[1,N_2]}} (\alpha_j - \alpha_{j-1})(\beta_l - \beta_{l-1})$$

Let us replace this equality in Eq. (5). It gives

$$Pl_{1\otimes 2}(\{x\}) = \sum_{\substack{F_i,G_k\\ i\in[1,L],k\in[1,K]}} m(F_i)m(G_k) \sum_{\substack{x\in F_{i,j}\cap G_{k,l}\\ j\in[1,N_1],l\in[1,N_2]}} (\alpha^i_j - \alpha^i_{j-1})(\beta^k_l - \beta^k_{l-1}),$$

which is indeed equivalent to equation (6). This finishes the proof. ■

This proposition tells us that, in the case of fuzzy belief structures, combining separately probabilistic information and fuzzy information with fusion rules corresponding to an assumption of independence can also be interpreted as a trace of the information resulting from the application of Dempster's rule to the mapped belief structure. Note also that, by using Dempster's rule, we keep all of its properties, such as associativity or commutativity, with or without normalisation. This is not necessarily the case when mixing Dempster's rule with fuzzy t-norms [17], even if both fusion rules considered separately are associative and commutative.

Note that, as in the case of single fuzzy sets, the equality only holds for singletons, since Demspter's rule applies to elements of the power set $2^{|\mathcal{X}|}$ of $\mathcal{X}$, while fuzzy combination rules apply to elements of $\mathcal{X}$. This is illustrated in the following example.

Example 2. Consider the two fuzzy belief structures $\tilde{m}_1, \tilde{m}_2$ of Example 1 and their mappings into belief structures m_1 and m_2. The result $m_{1\otimes 2}$ of applying Dempster's rule to m_1, m_2 is such that

$$m_{1\otimes 2}(\{x_2\}) = 0.32, \quad m_{1\otimes 2}(\{x_2, x_3\}) = 0.52,$$

$$m_{1\otimes 2}(\{x_1, x_2\}) = 0.08, \quad m_{1\otimes 2}(\mathcal{X}) = 0.08.$$

While considering the product of $\tilde{m}_1, \tilde{m}_2$ and the combination of the fuzzy focal elements by the product t-norm gives the fuzzy belief structure $\tilde{m}_{1\odot 2}$ with fuzzy focal elements F_{11} and F_{21} such that

	x_1	x_2	x_3
$F_{11} = \mu_{F_1}\mu_{G_1}$	0.2	1	0.5
$F_{21} = \mu_{F_2}\mu_{G_1}$	0	1	1

The mapping of $\tilde{m}_{1\odot 2}$ into a classical belief structure $m_{1\odot 2}$ gives

$$m_{1\odot 2}(\{x_2\}) = 0.4, \quad m_{1\odot 2}(\{x_2, x_3\}) = 0.44,$$

$$m_{1\odot 2}(\mathcal{X}) = 0.16,$$

and we do have

$$Pl_{1\odot 2}(\{x_1\}) = Pl_{1\otimes 2}(\{x_1\}) = 0.16,$$
$$Pl_{1\odot 2}(\{x_2\}) = Pl_{1\otimes 2}(\{x_2\}) = 1,$$
$$Pl_{1\odot 2}(\{x_3\}) = Pl_{1\otimes 2}(\{x_3\}) = 0.6,$$

but we have, for the event $\{x_1, x_3\}$,

$$Pl_{1\odot 2}(\{x_1, x_3\}) = 0.6 \neq Pl_{1\otimes 2}(\{x_1, x_3\}) = 0.68.$$

Note that as in the case of single fuzzy sets [23], the combination $m_{1\odot 2}$, which is computationally easier to achieve, is an inner approximation of $m_{1\otimes 2}$, in the sense that for any event $A \subseteq \mathcal{X}$, we have $Pl_{1\odot 2}(A) \leq Pl_{1\otimes 2}(A)$.

It is more difficult to interpret other rules that consider the product of probabilistic information and the combination of fuzzy information with other t-norms, for the reason that such combinations mix different independence assumptions (see, for instance [24]). Similarly, it may be difficult to interpret other fusion rules proper to belief structures (see, for example, recent propositions of Denoeux [22]) in terms of separate merging of fuzzy and probabilistic information. On the other hand, there may be situations where other rules than Dempster's rule could be preferred and justified, in order preserve a particular arrangement of the focal elements. For example, in order to extend bayesian inference, Walley [25] considers partially consonant belief functions (i.e., belief functions for which focal elements form non-overlapping groups of nested foci) and proposes a fusion rule, based on commonality measures, that merge two partially consonant belief functions into a belief function that is again partially consonant.

IV. Propagation

Let us now investigate the problem of propagating information through a function f whose variable uncertainty is described by fuzzy belief structures. As recalled in the introduction, these fuzzy belief structures could be the result of a fusion process, a statistical counting of imprecise observation, or a previous propagation (in the case of cascading propagation through multiple functions). Let $X^1, \ldots, X^M$ be M variables assuming their values on spaces $\mathcal{X}^1, \ldots, \mathcal{X}^M$ and $f : \times_{i=1}^M \mathcal{X}^i \to \mathcal{Y}$ a function from the Cartesian product of $\mathcal{X}^1, \ldots, \mathcal{X}^M$ to $\mathcal{Y}$. Propagation consists in evaluating the uncertainty on variable Y by propagating the uncertainty of $X^1, \ldots, X^M$ through f.

If uncertainty is directly given as a joint fuzzy belief structure $\tilde{m}$ over $\times_{i=1}^M \mathcal{X}_i$, with m its mapping in a belief structure, then if $E_1, \ldots, E_L$ are the focal elements of m, the propagated belief structure is such that, for $i = 1, \ldots, L$

$$E_i^Y = f(E_i) = \{f(\mathbf{x}) \in \mathcal{Y} | \mathbf{x} \in E_i\},$$
$$m(E_i^Y) = m(E_i),$$

where $\mathbf{x}$ denotes a vector of $\times_{i=1}^M \mathcal{X}^i$. However, such an ideal situation is unlikely to happen in practice, simply because giving directly a joint uncertainty representation is too difficult. The most common situation consists in combining individual uncertainty representations (here, fuzzy belief structures) given for each variable $\mathcal{X}^1, \ldots, \mathcal{X}^M$ into a joint model over $\times_{i=1}^M \mathcal{X}^i$ by using specific (in)dependence assumptions.

Before going into further details about how to combine and propagate fuzzy belief structures, let us first recall the classical independence assumptions used to respectively combine and propagate fuzzy sets and belief structures.

A. classical independence assumptions for fuzzy sets and belief structures

Zadeh's extension principle [26] is usually considered to propagate fuzzy sets. Given M fuzzy sets $F^1,\dots,F^M$ respectively defined over $\mathcal{X}^1,\dots,\mathcal{X}^M$, the fuzzy set F_f obtained by propagating them through f by the extension principle is such that, for any $y\in\mathcal{Y}$,

$$\mu_{F_f}(y)=\sup_{\substack{x_1,\dots,x_N\in\times_{i=1}^M\mathcal{X}^i\\ f(x_1,\dots,x_N)=y}}\min_{i=1,\dots,M}(\mu_{F^i}(x_i)).$$

This extension principle is equivalent to consider the joint fuzzy set $F^{1,M}$ over $\times_{i=1}^M\mathcal{X}_i$ such that $\mu_{F^{1,M}}(x_1,\dots,x_N)=\min_{i=1,\dots,M}(\mu_{F^i}(x_i))$ and then to propagate the classical consonant belief structure induced by $F^{1,M}$. In the sequel, we will call such a joint fuzzy set *non-interactive*, and will talk of non-interactivity for the associated independence notion. Note that an important advantage of the extension principle, from a practical standpoint, is that it is computationally attractive, since it comes down to apply interval analysis at each α-cut level.

Now, when uncertainty is described by M belief structures $m^1,\dots,m^M$ having respectively $L_1,\dots,L_M$ focal elements, the classical independence notion used to propagate them is the random set independence notion [27]. That is, for any set $E\subseteq\times_{i=1}^M\mathcal{X}_i$,

$$m_{RI}(E)=\sum_{\substack{\times_{i=1}^M E_l^i=E\\ l\in\{1,\dots,L_i\}}} m(E_l^i),$$

with E_l^i the l^{th} focal element of m^i.

The assumption of random set independence can be interpreted as the assumption that the sources of information of each variable x^i are independent (e.g. different sensors measure each variable x_i). Also, an assumption of random set independence is conservative when compared to other notions of independence [28], and can thus be used as a conservative tool to approximate such assumptions (which are often difficult to handle in practice).

B. independence assumptions for fuzzy belief structures

Let us now consider that uncertainty over variables $X^1,\dots,X^M$ is described by fuzzy belief structures $\tilde{m}^1,\dots,\tilde{m}^M$ having respectively $L_1,\dots,L_M$ fuzzy focal elements. A classical assumption of independence [27] when dealing with this type of mixed probabilistic and fuzzy information is to assume stochastic independence for the probabilistic information and non-interaction for the fuzzy information. Under this assumption, if we consider the set $F_{i_1}^1,\dots,F_{i_M}^M$ of fuzzy focal elements and that $\alpha_1<\dots<\alpha_N$ are the distinct values that can take every fuzzy focal element of $\tilde{m}^1,\dots,\tilde{m}^M$, then for a given α_j the set $\times_{k=1}^M F_{i_k,\alpha_j}^k$ receive the mass

$$m_{SINI}(\times_{k=1}^M F_{i_k,j}^k)=(\alpha_j-\alpha_{j-1})\prod_{k=1}^M m(F_{i_k}^k),$$

with $F_{i_k,j}^k$ the α_j-cut of $F_{i_k}^k$. The belief structure m_{SINI} (for Stochastic Independence and Non Interaction) can then be propagated through the function f. This particular independence assumption reduces to the extension principle when structures are reduced to single fuzzy sets, and to the random set independence assumption when all focal elements are classical sets. However, this independence assumption clearly requires that the fuzzy and probabilistic information can be considered and treated separately. That such an assumption holds when fuzzy belief structures describe the global uncertainty about variables is questionable. For example, consider the case where the initial uncertainty is propagated through a function f_1, and that the fuzzy belief structure resulting from the propagation must again be combined with other ones and be propagated through a second function f_2. That a SINI assumption holds for this second propagation is doubtful, even if it held for the first one.

Instead of considering such a mixed independence assumption, one can consider the random set independence assumption between the belief structures $m_1,\dots,m_M$ mapped from $\tilde{m}^1,\dots,\tilde{m}^M$, with all advantages associated to such an assumption (approximation of other independence notions, clear generalisation of stochastic independence between variables).

In such a case, there is no correlation between α-cuts of different fuzzy focal elements, and for a set $F_{i_1}^1,\dots,F_{i_M}^M$ of fuzzy focal elements and given α_{j_k} levels for each of them, the set $\times_{k=1}^M F_{i_k,j_k}^k$ receive the mass

$$m_{RI}(\times_{k=1}^M F_{i_k,j_k}^k)=\prod_{k=1}^M m(F_{i_k}^k)(\alpha_{j_k}-\alpha_{j_k-1}).$$

This mass can then be propagated through function f. Note that in general none of the joint uncertainty models described by this two independence assumptions (SINI and RI) is included in the other [29], as we may have two events $A,B\subseteq\times_{i=1}^M\mathcal{X}_i$ for which $Pl_{RI}(A)\le Pl_{RINI}(A)$ and $Pl_{RI}(B)\ge Pl_{RINI}(B)$, with Pl_{RINI} and Pl_{RI} the plausibility measures induced by the two independence assumptions.

Example 3. Consider two variables X_1,X_2 whose uncertainty is respectively described by fuzzy belief structures $\tilde{m}_1,\tilde{m}_2$ of Example 1. We consider the simple function $f(Y)=X_1-X_2$. The joint belief structure m_{RINI} obtained by a combination of fuzzy non-interaction and random set independence is such that

$$m_{SINI}([1,9]\times[1,9])=0.16,\quad m_{SINI}([1,9]\times[3,9])=0.24,$$

$$m_{SINI}([1,6]\times[3,9])=0.4,\quad m_{SINI}([3,9]\times[1,9])=0.04,$$

$$m_{SINI}([3,9]\times[3,9])=0.16,$$

and the belief structure m_{RI} obtained by an assumption of random set independence between m_1,m_2 is such that

$$m_{SINI}([1,9]\times[1,9])=0.08,\quad m_{SINI}([1,9]\times[3,9])=0.32,$$

$$m_{SINI}([1,6]\times[1,9])=0.08,\quad m_{SINI}([1,6]\times[3,9])=0.32,$$

$m_{SINI}([3,9]\times[1,9]) = 0.04, \quad m_{SINI}([3,9]\times[3,9]) = 0.16.$

The results of the propagation of both belief structures through f is summarized in the following table

Set on $\mathcal{Y}$	m_{SINI}	m_{RI}
$[-8,8]$	0.08	0.16
$[-8,6]$	0.32	0.24
$[-8,5]$	0.08	0
$[-8,3]$	0.32	0.4
$[-6,8]$	0.04	0.04
$[-6,6]$	0.16	0.16

And we have $Bel_{RI}([-8,3]) = 0.32 < Bel_{SINI}([-8,3]) = 0.4$, while $Bel_{RI}([-8,6]) = 0.88 > Bel_{SINI}([-8,6]) = 0.8$.

Note that, if one wants to keep the computational advantages associated to the use of the fuzzy extension principle while outer approximating the result that would give an assumption of random set independence, it is possible to apply a simple transformation to fuzzy focal elements in order to obtain such an outer approximation [30]. Namely, given the number M of dimensions, it comes down to transform any fuzzy focal set F_i into F'_i such that for any $x \in \mathcal{X}$

$$F'_i(x) = (-1)^{M+1}(F_i(x) - 1)^M + 1.$$

Should such a transformation be applied to alleviate the computational burden, it must be kept in mind that the related information loss can be important.

V. Decision

The problem of decision making with belief structures is still an active research topic [31]. Here, we consider that the decision problem consists in choosing among a set of acts $f_1, \ldots, f_D$ given some uncertainty over a variable X represented by a fuzzy belief structure $\tilde{m}$. To each of these acts corresponds a (bounded) real-valued utility function $f_i : \mathcal{X} \to \mathbb{R}$, with $f_i(x)$ representing the benefit (or loss) of choosing f_i when x is the true value of X. When manipulating imprecise uncertainty representations such as belief structures, there are two main strategies to extend classical decision rules (here, expected utility): use the imprecise information to select a single optimal act, or consider that the presence of imprecision only allows us to select a subgroup of possibly optimal acts. For each strategies, we will explore one of the main proposition proposed in the literature that can be applied to belief structure.

We will first interest ourselves to Smet's pignistic probability [32], which consists in transforming a belief structure into a single probability (i.e. the gravity center of the probability set it represents). We will then interest ourselves to Walley's maximality criterion, proposed in the framework of imprecise probability theory. This criterion consists in searching optimal (i.e., non-dominated) acts, possibly ending with a set of possible choices. For each decision rule, we will consider its formula for belief structures issued from fuzzy belief structures, and relate it to the original fuzzy belief structure.

A. Pignistic probability

Given a belief structure m defined on $\mathcal{X}$ and with focal sets $E_1, \ldots, E_L$, the pignistic transformation consists in mapping m to a probability distribution $BetP_m : \mathcal{X} \to [0,1]$ such that, for any $x \in \mathcal{X}$, $BetP_m(x) = \sum_{\{E_i | x \in \underline{E_i}\}} {}^{m(E_i)}/_{|E_i|}$, with $|E_i|$ the cardinality of E_i. The optimal act $\overline{f}$ can then be chosen as the one maximising expected utility under $BetP_m$, i.e.

$$\overline{f} = \arg \max_{f_1,\ldots,f_D} \sum_{x \in \mathcal{X}} f_i(x) BetP_m(x).$$

Using the pignistic probability has been justified by Smets in the framework of the TBM [32]. It is also equivalent to the Shapley value in game theory, and corresponds to the gravity center of $\mathcal{P}_m$, the probability set induced by m (a parallel can then be done with mechanic, where representing a set by its gravity centre is usual).

When we map a fuzzy belief structure $\tilde{m}$ with focal elements $F_1, \ldots, F_L$ into the belief structure m, the pignistic probability $BetP_m$ of an element $x \in \mathcal{X}$ is given by the formula

$$BetP_m(x) = \sum_{x \in F_{i,j}} \frac{m(F_i)(\alpha_j - \alpha_{j-1})}{|F_{i,j}|}. \qquad (7)$$

With $|F_{i,j}|$ the cardinality of the α_j-cut of F_i. Note that this generalisation of the pignistic probability is different from other propositions [1], where

$$BetP'_m(x) = \sum_{i=1}^{L} m(F_i) \left(\frac{\mu_{F_i}(x)}{\sum_{x \in \mathcal{X}} \mu_{F_i}(x)} \right). \qquad (8)$$

The next proposition shows that Equation (7) is a meaningful extension of the pignistic transform to fuzzy belief structure, in the sense that it can be interpreted as a weighted mean of pignistic transforms done on each fuzzy focal element.

Proposition 2. *Let $\tilde{m}$ be a fuzzy belief structure with $F_1, \ldots, F_L$ its focal elements and m its mapping into a belief structure. Let $BetP_{\tilde{m},i}$ denote the pignistic probability induced by F_i, and $BetP_m$ the pignistic transform of m. Then, the following equality holds for any $x \in \mathcal{X}$*

$$BetP_m(x) = \sum_{i=1}^{L} m(F_i) BetP_{\tilde{m},i}(x)$$

Proof: Let $\alpha_0 = 0 < \alpha_1 < \ldots < \alpha_N$ be the distinct function membership values of fuzzy focal elements F_I. For a given fuzzy focal element F_i, $BetP_{\tilde{m},i}(x)$ reads

$$BetP_{\tilde{m},i}(x) = \sum_{j=1}^{N} \frac{(\alpha_j - \alpha_{j-1})}{|F_{i,j}|}.$$

If we denote by $BetP_{\tilde{m}}$ as the mean of these probabilities weighted by focal elements masses, we have

$$\begin{aligned} BetP_{\tilde{m}}(x) &= \sum_{i=1}^{L} m(F_i) BetP_{\tilde{m},i}(x) \\ &= \sum_{i=1}^{L} m(F_i) \sum_{j=1}^{N} \frac{(\alpha_j - \alpha_{j-1})}{|F_{i,j}|}, \end{aligned}$$

which is equal to Eq (7). ■

Eq (7) then simply comes down to take the mean of the gravity centres of each $\mathcal{P}_{F_i}$, and is well in accordance with the pignistic approach recommended by Smets. The extension given by Eq. (8) is less coherent with a pignistic approach, since it considers the mean of probabilities obtained by applying the plausibility transform [33] to each fuzzy focal element F_i (note that, in this case, the resulting probability may not even be in $\mathcal{P}_m$).

B. Maximality criterion

Pignistic transform maps an imprecise uncertainty representation to a probability (hence precise) uncertainty representation, allowing to use the classical expected utility criterion to choose among acts. Another argument, whose main proponents work in the framework of imprecise probability theory [34], is that imprecise uncertainty representations could generate imprecise decisions, our limited information only allowing us, in some situations, to pinpoint a set of optimal acts rather than a single one.

First, let us recall the notion of lower and upper expected values of a function f over a probability set $\mathcal{P}$: given a (utility) function $f : \mathcal{X} \to \mathbb{R}$, its lower and upper expected vaues over $\mathcal{P}$, respectively denoted by $\underline{\mathbb{E}}_{\mathcal{P}}(f)$ and $\overline{\mathbb{E}}_{\mathcal{P}}(f)$, are such that

$$\underline{\mathbb{E}}_{\mathcal{P}}(f) = \inf_{p \in \mathcal{P}} \mathbb{E}_p(f) \quad \overline{\mathbb{E}}_{\mathcal{P}}(f) = \sup_{p \in \mathcal{P}} \mathbb{E}_p(f),$$

where $\mathbb{E}_p$ is the classical expectation of f given p. Note that both upper and lower expectations coincide and reduce to classical expectation when $\mathcal{P} = \{p\}$ is reduced to a singleton.

Given acts $f_1, \ldots, f_D$ and a probability set $\mathcal{P}$, the maximality criterion consists in inducing a partial order $\succeq_{\mathcal{M}}$ among acts such that $f_i \succeq_{\mathcal{M}} f_j$ if and only if $\underline{\mathbb{E}}_{\mathcal{P}}(f_i - f_j) \geq 0$, meaning that given our knowledge about X, there is some benefit in exchanging f_j for f_i. The set $\mathcal{M}_{\mathcal{P}}$ of optimal acts under maximality criterion is then

$$\mathcal{M}_{\mathcal{P}} = \{f_i | \nexists f_j \ s.t. \ f_j \succeq_{\mathcal{M}} f_i\}$$

Let $\mathcal{P}_m$ be the probability set induced by the belief structure m, itself mapped from the fuzzy belief structure $\tilde{m}$. The following proposition shows that mapping $\tilde{m}$ into a belief structure m is also coherent with maximality.

Proposition 3. *Let $\mathcal{P}_{F_i}$ be the probability set induced by the fuzzy focal element F_i, then, for two acts f_i, f_j*

$$\underline{\mathbb{E}}_{\mathcal{P}_m}(f_i - f_j) = \sum_{i=1}^{N} m(F_i) \underline{\mathbb{E}}_{\mathcal{P}_{F_i}}(f_i - f_j)$$

Proof: Immediate, given relation (4) of Section II and the fact that the lower expectation of a convex sum of probability sets is equal to the convex sum of the lower expectation of each of these probability set. ■

This proposition indicates that $f_i \succeq_{\mathcal{M}} f_j$ under uncertainty m if and only there is a positive average benefit to exchange f_i for f_j on each of the fuzzy focal element. Therefore, usual decision rules used with imprecise uncertainty representations are coherent with the mapping of a fuzzy belief structure $\tilde{m}$ into a belief structure m, be it in the setting of the TBM or of imprecise probability theory.

VI. Conclusion

In this paper, we have studied different operations on fuzzy belief structures when they are mapped into belief structures containing the same information. Such a mapping allows to use tools proper to belief structures rather than considering mixing of fuzzy and probabilistic calculi. Such operations have the advantage that they often have a clearer interpretation then mixed operations, making it more easy to identify whether they apply to a given problem.

For instance, when merging multiple fuzzy belief structures, one can safely come back to the classical Dempser's rule, as Proposition 1 indicates that this rule captures the information resulting from independence assumptions made separately on the fuzzy focal elements and on their weights. Establishing similar links with other rules appears more difficult, as interpreting such rules is already a bit tricky when they are applied to belief structures. The same conclusions apply to uncertainty propagation, as checking or assuming random set independence between some variables appear more feasible than mixing fuzzy non-interactivity and random set independence notions.

We have also checked that decision rules remain coherent when considering the mapping of a fuzzy belief structure into a belief structure.

Note that this mapping essentially makes sense when the fuzzy belief structure represent our uncertainty about the true value of a variable X. There are other situations, for example in those cases where the domain elements are themseles fuzzy, or when the fuzzy sets describe gradual notions, where transforming a fuzzy belief structure into a belief structure makes less sense.

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