International Journal of General Systems
Vol. 00, No. 00, Month 200x, 1–22

RESEARCH ARTICLE

Numerical accuracy and efficiency in the propagation of epistemic and aleatory uncertainty

Eric Chojnacki and Jean Baccou and Sébastien Destercke*

Institut de radioprotection et de sûreté nucléaire (IRSN)
13115 St-Paul Lez Durance, France

(*Received 00 Month 200x; final version received 00 Month 200x*)

The need to differentiate between epistemic and aleatory uncertainty is now well admitted by the risk analysis community. One way to do so is to model aleatory uncertainty by classical probability distributions and epistemic uncertainty by means of possibility distributions, and then propagate them by their respective calculus. The result of this propagation is a random fuzzy variable. When dealing with complex models, the computational cost of such a propagation quickly becomes too high. In this paper, we propose a numerical approach, the RaFu method, whose aim is to determine an optimal numerical strategy so that computational costs are reduced to their minimum, while using the theoretical framework mentioned above. We also give some means to take account of the resulting numerical error. The benefits of the RaFu method are shown by comparisons with previous methodologies.

Keywords: order statistics; epistemic uncertainty; sampling method; risk analysis; hybrid calculus

1. Introduction

Taking uncertainties into account has become of prime importance in many industrial applications. It is particularly true in safety studies, where misleading representations of uncertainties can lead to incautious and therefore potentially dangerous decisions.

Nowadays, a large majority of uncertainty analysts uses probabilistic models to represent uncertainties and Monte-Carlo simulations to propagate them through a model. In such approaches, both aleatory uncertainties (i.e due to the natural variability or randomness of an observed phenomenon) and epistemic uncertainties (i.e. due to the imprecision or poverty of available information) are modeled by probabilities. However, many arguments (Walley 1991, Helton and Oberkampf 2004, Ferson and Ginzburg 1996) converge to the conclusions that classical probabilities cannot adequately model epistemic uncertainties.

Therefore, recent works (Helton and Oberkampf 2004) have focused on methodologies able to handle both aleatory and epistemic uncertainties in an unified framework. One such method, proposed and justified by various authors (Bardossy and Fodor 2004, Ferson *et al.* 2003, Baudrit *et al.* 2006), consists in mixing probabilistic convolution (for aleatory uncertainty) with fuzzy calculus (for epistemic uncertainty) to model and propagate uncertainties. This theoretical approach, often referred to as hybrid approach, is the one considered here. Recent works (Baudrit *et al.* 2008) show that such methods provides results different from classical

*Corresponding author. Email: sdestercke@gmail.com

ISSN: 0308-1079 print/ISSN 1563-5104 online

DOI: 10.1080/0308107YYxxxxxxxx
http://www.informaworld.com

2-D Monte-Carlo simulations, usually used to differentiate between aleatory and epistemic uncertainties in classical probabilistic framework.

As propagating uncertainties with this approach often involves high computational costs, its domain of application has been limited so far to relatively simple models. In order to apply it to fields (such as nuclear safety) where models can be very complex and where computation costs constitute an important issue, more efficient propagation methods are needed.

This is why we propose in this work a new numerical method to propagate uncertainties with the above methodology. This method, based on sampling techniques, intends to reduce computational costs. It also allows one to address numerical accuracy issues, by using convergence results of Monte-Carlo methods and notions of order statistics (Lecoutre and Tassi 1987, Conover 1999). The key point of the method lies in the pre-processing of information related to the final desired result of the propagation, rather than post-processing it (as usually suggested).

Although this paper focuses on issues regarding safety studies, and thus on the estimation of uncertainties concerning threshold exceeding (i.e. cumulative distribution and so-called survival functions), the method presented here is not confined to such type of information.

This paper is organized as follows: Section 2 recalls theoretical bases used for representing and propagating aleatory and epistemic uncertainties in hybrid methodologies. The resulting output is no longer a random variable (as in classical probabilistic modelling) but a random fuzzy variable. In Section 3, we recall existing post-processing methods that extract relevant information from the model output, and discuss their computational cost. Section 4 introduces the proposed numerical treatment of aleatory and epistemic uncertainties (called the **RaFu** method, RaFu standing for Random/Fuzzy), that improves computational efficiency by avoiding the construction of the whole random fuzzy variable when possible. Finally, the RaFu method is illustrated on a simplified application in Section 5.

2. Representation and propagation of aleatory and epistemic uncertainties

In this section, we first recall basics about probability and possibility theories, the former being used to represent aleatory uncertainty, and the latter to represent epistemic uncertainty. Then, we explain how these two types of uncertainties are propagated through a model into a random fuzzy variable. Since our work focuses on the numerical treatment of hybrid-type approach (i.e. combining probability and possibility calculi), we do not intend to deeply discuss the advantages and limits of the two uncertainty theories. We refer to related works (Bardossy and Fodor 2004, Ferson *et al.* 2003, Baudrit *et al.* 2006) for detailed discussions about theoretical justifications.

2.1 *Representing uncertainty*

As mentioned previously, one can distinguish two main kinds of uncertainty. Aleatory uncertainty is due to the natural variability or randomness of an observed phenomenon. It can be, for instance, the variability inside a given population (e.g. gaussian distribution to describe the weight of a given nationality, exponential distribution corresponding to time failures of some class of components) or the variability of observed outcomes for a particular situation (e.g. dice tossing).

Epistemic uncertainty results from a lack of knowledge, of information. It can come from systematical error (e.g. a measurement which is not fully reliable), from

a poor quantity of data or from subjective uncertainty (e.g. an expert providing imprecise valued quantities).

Recent works (Walley 1991, Helton and Oberkampf 2004, Ferson and Ginzburg 1996) have shown that classical probabilities tend to confuse the two kinds of uncertainty and are not tailored to properly handle both of them. Other or more general frameworks thus need to be developed to separately treat both uncertainties. As already mentioned, we consider here that aleatory uncertainty is modeled and propagated by using classical probability theory (Feller 1971), while epistemic uncertainty is modeled and propagated with the help of possibility theory (Dubois and Prade 1988).

2.1.1 Aleatory uncertainty and probability theory

Given a probability space $(\Omega, \mathcal{F}, P)$, a probability measure P is defined as a mapping from $\mathcal{F}$ to $[0,1]$, such that $P(\Omega) = 1, P(\emptyset) = 0$ and for all $A, B \in \mathcal{F}$, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Here, we consider that $\mathcal{F}$ is either the power set of Ω (when Ω is discrete) or the Borel Algebra when $\Omega = \mathbb{R}$, the real line, therefore we will not mention $\mathcal{F}$ further on. From a probability measure P, we can define its probability distribution function p as the mapping from the sample space Ω (e.g. 1 to 6 in the case of a dice) to $[0,1]$ such that for any $\omega \in \Omega$, $p(\omega) = P(\{\omega\})$. For any subset $A \subseteq \Omega$, the probability measure is retrieved by

$$P(A) = \sum_{w \in A} p(w) \ \forall A \text{ measurable, } \text{ (discrete case) },$$

$$P(A) = \int_A p(w)dw \ \forall A \text{ measurable, } \text{ (continuous case)}$$

and $P(A)$ measures the likelihood of the event A

If X is a real random variable associated to P, the cumulative distribution function of X is a mapping $F_X : \mathbb{R} \to [0,1]$ defined for all $x \in \mathbb{R}$ as

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} p(w)dw$$

and which has a quasi-inverse given by F_X^{-1}. If α is an uniform random variable on $[0,1]$, then it is well known that the random variable $X = F_X^{-1}(\alpha)$ is distributed according to F_X.

This means that we can simulate a random variable X by simulating an uniform law on $[0,1]$ and associate to each sampled value α_i the corresponding element $x = F_X^{-1}(\alpha_i)$.

2.1.2 Epistemic uncertainty and possibility theory

Imprecise knowledge about a variable having a precise value can be described by the means of possibility theory (Dubois *et al.* 2000) . In particular, possibility distributions are well fitted to represent information about a variable given in terms of nested confidence intervals (a natural way to express uncertainty about variables, already considered by Cox (Cox 1958) and Birnbaum (Birnbaum 1961)). A possibility distribution is defined as a mapping $\pi : \Omega \to [0,1]$ which is here upper semi-continuous and normalized ($\exists x \in \Omega$ s.t. $\pi(x) = 1$). It is formally equivalent to the fuzzy set $\mu(x) = \pi(x)$. Distribution π describes the more or less plausible values of some uncertain variable X. To a possibility distribution are associated

two measures, namely the possibility (Π) and necessity (N) measures, which read:

$$\Pi(A) = \sup_{x \in A} \pi(x) \quad N(A) = \inf_{x \notin A} (1 - \pi(x))$$

The possibility measure indicates to which extent the event A is plausible, while the necessity measure indicates to which extent it is certain. They are dual, in the sense that $\Pi(A) = 1 - N(\overline{A})$, with $\overline{A}$ the complement of A. They obey the following axioms:

$$\Pi(A \cup B) = \max(\Pi(A), \Pi(B)) \quad N(A \cap B) = \min(N(A), N(B))$$

An α-cut of π is the interval $[\underline{x}_\alpha, \overline{x}_\alpha] = \{x, \pi(x) \geq \alpha\}$. The degree of certainty that $[\underline{x}_\alpha, \overline{x}_\alpha]$ contains the true value of X is $N([\underline{x}_\alpha, \overline{x}_\alpha]) = 1 - \alpha$. Conversely, a collection of nested sets A_i with (lower) confidence levels λ_i can be modeled as a possibility distribution, since the α-cut of a (continuous) possibility distribution can be understood as the probabilistic constraint $P(X \in [\underline{x}_\alpha, \overline{x}_\alpha]) \geq 1 - \alpha$, thus linking possibility distributions with imprecise probabilities (Dubois and Prade 1992, de Cooman and Aeyels 1999). In this setting, degrees of necessity are equated to lower probability bounds, and degrees of possibility to upper probability bounds.

As there is a one-to-one correspondence between levels $\alpha \in [0, 1]$ and the corresponding α-cut $[\underline{x}_\alpha, \overline{x}_\alpha]$, a possibility distribution can be simulated, similarly to probability distributions, by sampling values from an uniform law on $[0, 1]$ and by associating to each sampled value α_i the corresponding α-cut $[\underline{x}_{\alpha_i}, \overline{x}_{\alpha_i}]$.

2.2 *P-boxes*

The main question of safety studies is often to know, given uncertainties on inputs, whether or not the output value exceeds a given threshold. In a purely probabilistic framework, if the value of this threshold is x, the uncertainty on the exceeding of this threshold is given by the cumulative distribution function (CDF) $F(x) = P((-\infty, x])$.

If epistemic uncertainty is taken into account, the uncertainty over the exceeding of a threshold is no longer precise, and is given by a pair of lower and upper cumulative distribution functions $[\underline{F}, \overline{F}]$, usually called probability boxes (Ferson *et al.* 2003)[1] (p-boxes for short). The uncertainty on the exceeding of a threshold x is then expressed by a pair of values $[\underline{F}(x), \overline{F}(x)]$, bounding the potential values of $F(x) = P((-\infty, x])$. The width of the interval reflects our lack of information concerning some input parameters.

2.3 *Propagating both uncertainties into a random fuzzy variable*

Hybrid numbers (i.e. random fuzzy variables) as a means to express conjointly epistemic uncertainty and aleatory uncertainty were first proposed by Kaufmann and Gupta (Kaufmann and Gupta 1985). Latter on, methods based on this idea were proposed by Baudrit *et al.* (Baudrit *et al.* 2006), by Ferson and Ginzburg (Ferson and Ginzburg 1996) and by Cooper *et al.* (Cooper *et al.* 1996).

We consider that uncertainty bearing on input variables $X_1, \ldots, X_N$ has to be propagated through a model $Y = T(X_1, \ldots, X_N)$ with Y, the real-valued output.

[1]It must noted that, in the imprecise case, different sets of probabilities can be represented by the same p-box, whereas in the precise case, one cumulative distribution corresponds to one precise probability distribution, and inversely

We consider that $X_1, \ldots, X_k$ are random variables described by precise probability distributions $p_1, \ldots, p_k$, and $X_{k+1}, \ldots, X_N$ are fuzzy variables (i.e. imprecisely known variables) described by possibility distributions $\pi_{k+1}, \ldots, \pi_N$, all assuming values on the real line. Given this model, Kaufmann and Gupta originally proposed to propagate both types of uncertainty according to their respective calculus: probabilities by probabilistic convolution and possibility distributions by the means of extension principle (Dubois *et al.* 2000). When variables $X_1, \ldots, X_k$ take values $x_1, \ldots, x_k$, the extension principle reads, for any $y \in \mathbb{R}$

$$\pi^T(y) = \sup_{x_{k+1},\ldots,x_N, T(x_1,\ldots,x_N)=y} \min(\pi_{k+1}(x_{k+1}), \ldots, \pi_N(x_N)). \quad (1)$$

This extension principle extends classical interval computation in the following way: the distribution $\pi^T(y)$ can also be obtained by doing level-wise interval computation (Moore 1979, Jaulin *et al.* 2001), since we have

$$[\underline{y}_\alpha, \overline{y}_\alpha] = T(x_1, \ldots, x_k, [\underline{x}_\alpha, \overline{x}_\alpha]_{k+1}, \ldots, [\underline{x}_\alpha, \overline{x}_\alpha]_N), \ \forall \alpha \in [0,1] \quad (2)$$

This shows that extension principle assumes a complete correlation between α-cuts (i.e. between confidence levels), and does not generally encompass the result of classical probabilistic convolution. There exists other extensions of interval computations (Regan *et al.* 2004) proposing to deal with epistemic uncertainty by the means of imprecise probabilities. They usually provide more conservative results than the extension principle and, when applied to complex models, present a computational complexity even higher than the method considered here. Such extensions are not studied here. We also assume that dependencies between probability distributions $p_1, \ldots, p_k$ are well known, so that the joint distribution $p_{(1:k)}$ of $\times_{i=1,\ldots,k} X_i$ is well defined.

As finding the analytical and exact solution of the propagation is impossible in most situations, propagation is usually obtained by the following procedure:

(1) Generate M_p samples $x_{(1:k)_i} := \{x_{1,i}, \ldots, x_{k,i}\}$, $i = 1, \ldots, M_p$ stemming from the joint distribution $p_{(1:k)}$ of $\times_{i=1,\ldots,k} X_i$ by usual sampling techniques (Monte-Carlo, LHS, MCMC, ...)
(2) For each sample $x_{(1:k)_i}$, $i = 1, \ldots, M_p$, build a discretized approximation $\tilde{\pi}_i^T$ of the propagated possibility distribution π_i^T (see Equation (1)) by computing (2) for a finite collection $0 \leq \alpha_1 < \ldots < \alpha_{M_\pi} \leq 1$ of M_π α-cuts
(3) Assign a probability mass of $1/M_p$ to each obtained distribution $\tilde{\pi}_i^T$, $i = 1, \ldots, M_p$.

Values and intervals sampled from probability and possibility distributions are illustrated in Figure 1. The result of the whole procedure is an hybrid number, that is a probability distribution bearing on possibility distributions $\tilde{\pi}_i^T$, formally equivalent to a random fuzzy variable. It is illustrated in Figure 2. For simplicity of notation, we denote this random fuzzy variable, which describes our uncertainty on Y resulting from the propagation, by $(p_{(1:k)}, \tilde{\pi})^T$. Note that this procedure requires to achieve $M_p \times M_\pi$ interval propagations, with the value M_π being usually around 20.

3. Information extraction: existing post-processing techniques

Now that imprecision is explicitly modeled in our uncertainty representations, the probability of an event resulting from this propagation is no longer precise, but is

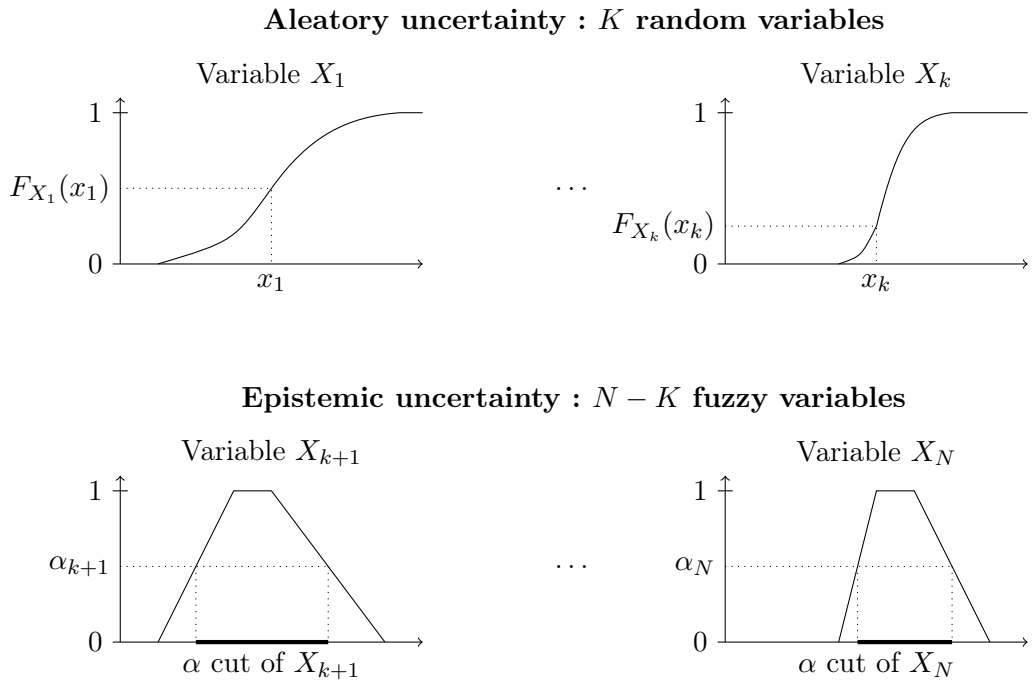

Figure 1. Sampling of random and fuzzy variables.

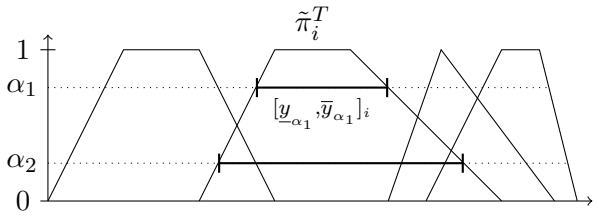

Figure 2. Random fuzzy variable.

instead delimited by lower and upper bounds. As analyzing the intrinsic information conveyed by the full random fuzzy variable is very difficult, it is necessary to propose some way to summarize or extract the useful information from the random fuzzy variable $(p_{(1:k)}, \tilde{\pi})^T$. For this reason, Ferson and Ginzburg (Ferson and Ginzburg 1996) and Baudrit *et al.* (Baudrit *et al.* 2006) have proposed different post-processing of $(p_{(1:k)}, \tilde{\pi})^T$ so that the resulting summary would be in the shape of one or multiple p-boxes.

Denote $[\underline{y}_\alpha, \overline{y}_\alpha]_i$ the α-cut of the i^{th} fuzzy set $\tilde{\pi}_i^T$, with $i = 1, \ldots, M_p$. For each value $\alpha \in [0, 1]$, we thus have a collection of M_p intervals. If we order and reindex the M_p values $\underline{y}_\alpha$ such that $\underline{y}_\alpha^i \leq \underline{y}_\alpha^j$ iff $i \leq j$, and assign to each of them a probability mass $1/M_p$, we can build the associated cumulative distribution function $\underline{F}_\alpha$ such that $\underline{F}_\alpha(\underline{y}_\alpha^i) = 1/M_p$. Upper values $\overline{y}^j$ can be treated likewise to obtain an upper distribution $\overline{F}_\alpha$. This can be done for every value $\alpha \in [0, 1]$, and since the α-cuts of a fuzzy set are nested, we have that $\underline{y}_\alpha^i \geq \underline{y}_\beta^i$ ($\overline{y}_\alpha^i \leq \overline{y}_\beta^i$) if $\alpha \geq \beta$, implying that $\underline{F}_\alpha(x) \geq \underline{F}_\beta(x)$ ($\overline{F}_\alpha(x) \leq \overline{F}_\beta(x)$) if $\alpha \geq \beta$. This shows how we can extract a collection of p-boxes $[\underline{F}_\alpha, \overline{F}_\alpha]$ from the random fuzzy variable $(p_{(1:k)}, \tilde{\pi})^T$ (see Figure 3).

Although the information conveyed by the collection of p-boxes $[\underline{F}_\alpha, \overline{F}_\alpha]$ is poorer than the information contained in the whole random fuzzy variable (information is

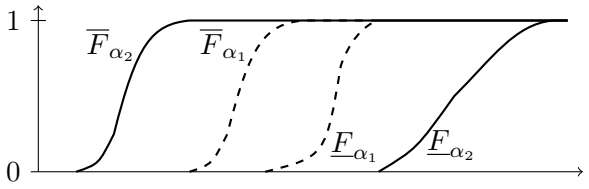

Figure 3. Pairs of lower and upper cumulative distribution functions extracted from the random fuzzy variable of figure 2 ($\alpha_1 \geq \alpha_2$).

lost by projecting the structure on events of the type $(-\infty, x]$), it is sufficient in most applications encountered in safety or reliability studies.

Nevertheless, the whole collection of p-boxes $[\underline{F}_\alpha, \overline{F}_\alpha]$ is still a complex representation, and in order to be useful to a decision maker, it should be summarized further. This is the objective of post-treatments recalled in the next section and proposed by Ferson and Ginzburg (Ferson and Ginzburg 1996) and by Baudrit *et al* (Baudrit *et al.* 2006).

3.1 *Ferson's post-treatment*

Ferson proposes to fix one or multiple confidence levels α and then to build the lower and upper cumulative distributions $[\underline{F}_\alpha, \overline{F}_\alpha]$ associated to this (these) particular value(s).

For example, choosing the value $\alpha = 1$ and the p-box $[\underline{F}_1, \overline{F}_1]$ corresponds to an "optimistic" behavior regarding epistemic uncertainty, since the imprecision of the result is minimized, while choosing the value $\alpha = 0$ and the p-box $[\underline{F}_0, \overline{F}_0]$ corresponds to "pessimistic" behavior, imprecision being maximized in this case. All other p-boxes $[\underline{F}_\alpha, \overline{F}_\alpha]$ are between these pairs and represent intermediate behavior (Figure 4).

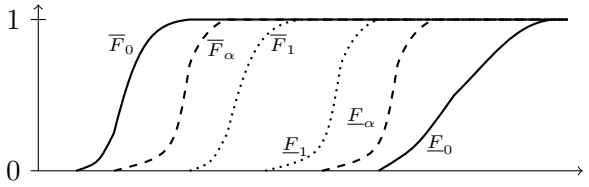

Figure 4. Pairs of lower and upper cumulative distribution functions associated to Ferson's post-treatment ($0<\alpha<1$).

Note that, even in the most optimistic case, there can still remain some imprecision, unless the α-cut of level 1 of every possibility distribution $\pi_{k+1}, \ldots, \pi_N$ is a single number (e.g. reference value, mode, ...).

3.2 *Baudrit et al.'s post-treatment*

In the post-processing of Baudrit *et al.*, called "homogeneous" post-processing, only one lower and one upper cumulative distributions, here respectively denoted $\underline{F}_{av}$ and $\overline{F}_{av}$, are built. The resulting p-box $[\underline{F}_{av}, \overline{F}_{av}]$ corresponds to the average taken over all p-boxes $[\underline{F}_\alpha, \overline{F}_\alpha]$, namely:

$$\underline{F}_{av} = \int_0^1 \underline{F}_\alpha d\alpha \qquad \overline{F}_{av} = \int_0^1 \overline{F}_\alpha d\alpha$$

and the p-box $[\underline{F}_{av}, \overline{F}_{av}]$ is always between the p-boxes $[\underline{F}_0, \overline{F}_0]$ and $[\underline{F}_1, \overline{F}_1]$ (Figure 5).

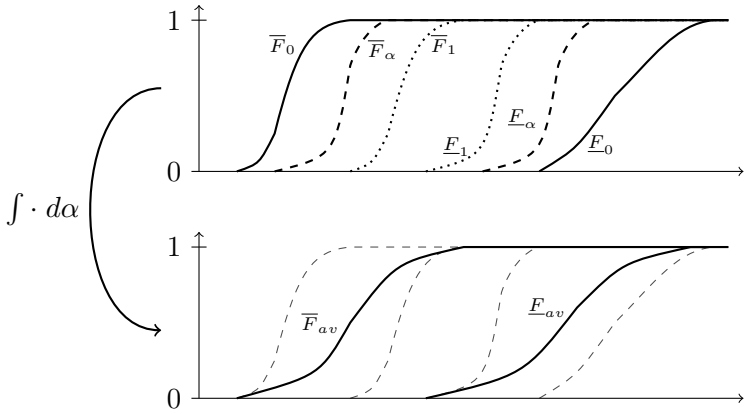

Figure 5. Pair of lower and upper cumulative distribution functions associated to Baudrit *et al.*'s post-treatment.

Both Ferson's and Baudrit *et al.*'s post-treatments require to first build the whole random fuzzy variable (as described in Section2.3). As mentioned before, this strategy can be computationally costly: let us suppose that $M_p = 100$ samplings are done on the k first parameters and that for each of them the corresponding fuzzy number is approximated by taking $M_\pi = 21$ α-cuts ($\alpha = (0, 0.05, ..., 1)$). Then, 2100 interval calculations are needed to build the final result.

In many applications, assuming one can afford so much computations is unrealistic. This is particularly true in fields such as nuclear safety, spatial exploration or aeronautics (Oberguggenberger *et al.* 2007), where very complex models are often used. Moreover, although they both propose to deal with complex models by using numerical discretization, neither Baudrit *et al.* nor Ferson consider the question of numerical accuracy. That is why we propose in the next section a new numerical method (called **RaFu** method, for Random/Fuzzy) that addresses the problem of evaluating numerical accuracy and allows one to reduce the number of required computation to reach a given result by pre-processing rather than post-processing a part of the information.

4. The RaFu method

The RaFu method uses the same theoretical framework as the one considered in Section 2. It intends to minimize the number of required computations to reach a given response. As pointed above, building the whole random fuzzy variable can be very costly and, in those situations where we are only interested in some specific features of the information contained in it, unnecessary.

Briefly, given the input distributions $p_1, \ldots, p_k, \pi_{k+1}, \ldots, \pi_N$ and a final desired response, the method consists in sampling from these distributions in an optimized way, so that a minimal amount of samples is propagated in order to reach the desired response with a given numerical accuracy. The method is fully detailed in the sequel.

$$\begin{pmatrix} F_{X_1}^{-1}(\alpha_{1,1}) & \cdots & F_{X_k}^{-1}(\alpha_{k,1}) & [\underline{x}_{\alpha_{k+1,1}}, \overline{x}_{\alpha_{k+1,1}}]_{k+1} & \cdots & [\underline{x}_{\alpha_{N,1}}, \overline{x}_{\alpha_{N,1}}]_N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ F_{X_1}^{-1}(\alpha_{1,i}) & \cdots & F_{X_k}^{-1}(\alpha_{k,i}) & [\underline{x}_{\alpha_{k+1,i}}, \overline{x}_{\alpha_{k+1,i}}]_{k+1} & \cdots & [\underline{x}_{\alpha_{N,i}}, \overline{x}_{\alpha_{N,i}}]_N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ F_{X_1}^{-1}(\alpha_{1,M}) & \cdots & F_{X_k}^{-1}(\alpha_{1,M}) & [\underline{x}_{\alpha_{k+1,M}}, \overline{x}_{\alpha_{k+1,M}}]_{k+1} & \cdots & [\underline{x}_{\alpha_{N,M}}, \overline{x}_{\alpha_{N,M}}]_N \end{pmatrix} \overset{T}{\Rightarrow} \begin{pmatrix} [\underline{y}, \overline{y}]_1 \\ \vdots \\ [\underline{y}, \overline{y}]_i \\ \vdots \\ [\underline{y}, \overline{y}]_M \end{pmatrix}$$

$$[\underline{y}, \overline{y}]_i = T(F_{X_1}^{-1}(\alpha_{1,i}), \ldots, F_{X_k}^{-1}(\alpha_{k,i}), [\underline{x}_{\alpha_{k+1,i}}, \overline{x}_{\alpha_{k+1,i}}]_{k+1}, \ldots, [\underline{x}_{\alpha_{N,i}}, \overline{x}_{\alpha_{N,i}}]_N)$$

Figure 6. Illustration of sample matrix

4.1 *Pre-processing rather than post-processing*

In practical applications, the quantity that has to be evaluated is often known before propagating. It can be the potential mean value of the output, the value of a particular percentile or the probability of exceeding a given value. In other cases, one has an idea about the behavior to adopt. For example, if the safety study concerns a critical issue (e.g. transfer of dangerous polluting elements), it is natural to adopt a very conservative and cautious attitude (i.e. in our case, an α close to zero), while in situations where the behavior to adopt is more ambiguous, one can choose to adopt a balanced attitude (i.e. use Baudrit *et al.*'s post-processing).

The key point of the RaFu method is to replace the classical post-processing step by a pre-processing one. In this pre-processing, the decision maker (DM)[1] provides a triplet of parameters $(\gamma_S, \gamma_E, \gamma_A)$ that corresponds to the quantity he's interested in and the numerical precision he wants to reach.

Once these parameters $(\gamma_S, \gamma_E, \gamma_A)$ have been specified, the RaFu method consists in building an optimal sampling strategy that allows one to reach the desired quantity with a minimal amount of calculations. This strategy corresponds to a number M of specific samples. Each sample consists of N values, the k first being single values sampled from $p_1, \ldots, p_k$ according to γ_S, while the $N-k$ other values are α-cuts (generally intervals) sampled from $\pi_{k+1}, \ldots, \pi_N$ according to parameter γ_E. The result, illustrated by Figure 6, is a matrix of M samples that must then be propagated through the model T.

4.1.1 Parameter γ_S

γ_S is the parameter related to the aleatory uncertainty. It consists of two sub-parameters:

- γ_{S_i} concerns the dependence structure between input variables $X_1, \ldots, X_k$ and determines how values are to be sampled from $p_1, \ldots, p_k$. In other words, it specifies the values $\alpha_{1,i}, \ldots, \alpha_{k,i}$, $i = 1, \ldots, M$ in Figure 6.
  Usual dependence structures that can be reproduced by numerical sampling and can be specified in γ_{S_i} are:
- stochastic independence between $X_1, \ldots, X_k$,
- rank correlations between variables (Iman and Conover 1982),
- specifications of copulas (Nelsen 2005),
- direct specification of joint distributions.

[1]This decision maker can assume many forms, it can be a committee, an official guideline, a single individual, ...

The k first values of each sample are then sampled (or reordered after sampling) accordingly to the specified dependence structure. Note that the computational cost of applying the above type of dependence structures is negligible, especially when compared to calculation time of complex models.

- γ_{S_o} concerns the stochastic quantity the DM is interested in. It provides which information must be extracted from propagated data $[\underline{y}, \overline{y}]_i$, $i = 1, \ldots, M$.

In safety studies, stochastic quantities that the DM usually wants to evaluate are typically:

- the mean and/or the variance ($\gamma_{S_o} := \{E(Y), V(Y)\}$),
- the value of a given percentile ($\gamma_{S_o} := \{q\%\}$),
- the uncertainty of exceeding a given threshold ($\gamma_{S_o} := \{F(x)\}$, with x the threshold value),
- the whole cumulative distribution function ($\gamma_{S_o} := \{F(x)\}\ \forall x$).

γ_S thus corresponds to information used in usual sampling methods where uncertainty is modeled entirely by classical probabilities.

4.1.2 Parameter γ_E

γ_E is the parameter related to epistemic uncertainty and to the DM behavior with regard to this uncertainty. It determines how α-cuts or values from $\pi_{k+1}, \ldots, \pi_N$ are to be sampled.

First, we assume (without loss of generality) that M_p values have been sampled from the joint distribution $p_{(1:k)}$ of $\times_{i=1,\ldots,k} X_i$, and we denote them $x_{(1:k)_i} := \{F^{-1}_{X_1}(\alpha_{1,i}), \ldots, F^{-1}_{X_k}(\alpha_{k,i})\}$, $i = 1, \ldots, M_p$. Then, typical strategies used in the RaFu method for the choice of parameter γ_E are, for instance:

Strategy 1 Fixed α ($\gamma_E := \{\alpha\}$): associate to any $x_{(1:k)_i}$ the cuts of fixed level α of distributions $\pi_{k+1}, \ldots, \pi_N$. This comes down to consider, in Figure 6, that $\alpha_{j,i} = \alpha$ for $j = k+1, \ldots, N$, $i = 1, \ldots, M_p$. For the DM, this choice is equivalent to adopt a given behavior with respect to epistemic uncertainty, ranging from total pessimism ($\gamma_E := \{\alpha = 0\}$) to total optimism ($\gamma_E := \{1\}$).

Strategy 2 Vector $\tilde{\alpha} = (\alpha^1, \ldots, \alpha^{M_\pi})$: duplicate the M_p samples $x_{(1:k)_i}$ M_π times, thus resulting in $M_p \times M_\pi$ samples $x_{(1:k)_{i,j}}$, $i = 1, \ldots, M_p$, $j = 1, \ldots, M_\pi$. For a fixed i, $x_{(1:k)_{i,j}}$ is constructed by associating to sample $x_{(1:k)_i}$ the α-cuts $[\underline{x}_{\alpha_{k+1,j}}, \overline{x}_{\alpha_{k+1,j}}]_{k+1}, \ldots, [\underline{x}_{\alpha_{N,j}}, \overline{x}_{\alpha_{N,j}}]_N$ with $\alpha_{k+1,j} = \ldots = \alpha_{N,j} = \alpha^j$, the j^{th} element of vector $\tilde{\alpha}$. This is equivalent to apply the previous strategy M_π times. For example, the vector of values can be the couple total pessimism/optimism ($\gamma_E := \{\ \alpha = (0, 1)\}$) or a given number of discretization steps of each distribution ($\gamma_E := \{\ \alpha = (0/n, \ldots, i/n, \ldots, n/n = 1)\}$, with n the number of steps). In this last case, we retrieve the usual propagation method described in Section 2.3.

Strategy 3 Partially randomized α: for each sample $x_{(1:k)_i}$, sample (independently) a value α^r from a uniform law on $[0, 1]$ and associate to $x_{(1:k)_i}$ the α-cuts $[\underline{x}_{\alpha_{k+1,i}}, \overline{x}_{\alpha_{k+1,i}}]_{k+1}, \ldots, [\underline{x}_{\alpha_{N,i}}, \overline{x}_{\alpha_{N,i}}]_N$ such that $\alpha_{k+1,i} = \ldots = \alpha_{N,i} = \alpha^r$. As we shall see later, this kind of sampling allows to "average" over all α-cuts.

Strategy 4 Totally randomized α: for each sample $x_{(1:k)_i}$, let $\alpha^{r_1}, \ldots, \alpha^{r_{N-k}}$ be $N - k$ values sampled from independent uniform laws on $[0, 1]$, this strategy associates to $x_{(1:k)_i}$ the cuts $[\underline{x}_{\alpha_{k+1,i}}, \overline{x}_{\alpha_{k+1,i}}]_{k+1}, \ldots, [\underline{x}_{\alpha_{N,i}}, \overline{x}_{\alpha_{N,i}}]_N$ with $\alpha_{k+1,i} = \alpha^{r_1}, \ldots, \alpha_{N,i} = \alpha^{r_{N-k}}$. This kind of sampling simulates the so-called notion of random

set independence in imprecise probabilities. It can be interpreted as an assumption of independence between the sources evaluating the epistemic uncertainties (sensors, experts, ...), or, if possibility distributions are to be interpreted as sets of probabilities (Dubois and Prade 1992), as a means to simulate stochastic independence among the probabilities in these sets (Couso *et al.* 2000). Some dependencies can also be assumed between the uniform laws on which α-cuts are sampled (Alvarez 2006). Nevertheless, how such dependency structures between possibility distributions can be interpreted is still unclear, and requires further research.

Each of the above strategies suggests different choices of $\{\alpha_{i,j}\}_{i=k+1,...,N;j=1,...,M}$ in matrix of Figure 6. The choice of one of them also influences the final number of samples to be propagated: this number will be M_p in the case of Strategies 1,3 and 4 and $M_p \times M_\pi$ in the case of Strategy 2 (where M_π is the number of different α-levels chosen by the DM). The strategy selection before the propagation is one of the main advantage of the RaFu method. In many situations, it leads to less propagation than the usual $M_p \times M_\pi$ propagations required to build the whole random fuzzy variable (see Section 2.3).

Note that, since Monte-Carlo sampling is primarily a numerical tool allowing to estimate complex integrals, any quantity that can be expressed in term of an integral over $p_1, \ldots, p_k, \pi_{k+1}, \ldots, \pi_N$ can, in principle, be estimated by the right sampling strategy.

Remark 1 : γ_{S_i} and γ_E define two separate "dependence" structures respectively related to aleatory and epistemic uncertainties. Considering dependencies between random and fuzzy variables still remains an open question and is not considered in this paper.

4.1.3 Parameter γ_A

Numerical approximation always means approximation error. One of the interest of the Monte-Carlo sampling is that convergence theorems allow one to quantify this error. Parameter γ_A is related to this numerical error and has a direct effect on the number M_p of samples to be propagated. It can be used in two ways: either the DM specifies a goal in term of numerical accuracy and the number of samples required to reach this accuracy is then determined, or the DM specifies a maximal number of samples that can be made (in accordance with available ressources), and the reachable numerical accuracy is evaluated.

Sometimes, it is possible to determine before propagating the required number of samples to reach a given accuracy, or the reachable accuracy with a given number of samples. In this case, the DM can fix his final choice before anything is done. When numerical accuracy can only be determined after the propagation, a simple strategy consists in making a first propagation with a low number of samples, and then to increase this number until the numerical accuracy satisfies the DM or until the maximal number of affordable propagations is reached. In this paper, we focus on a method evaluating numerical accuracy by the use of order statistics (Lecoutre and Tassi 1987, Conover 1999)[1], and which pertains to the cases where numerical accuracy can be determined beforehand.

Let us note X_q the q percentile of a random variable X. From a sample of size N, the use of order statistics consists in considering the ordered values $x_{(1)} \leq \ldots \leq x_{(N)}$ drawn from the random variable X. If the N values are drawn randomly and

[1]Often quoted as the use of Wilk's formula (Wilks 1962)

independently, the following equation

$$P(X_{(K)} < X_q) = \sum_{i=K}^{N} \left(\begin{array}{c} i \\ N \end{array} \right) q^i (1-q)^{N-i} \tag{3}$$

holds. This is equivalent to saying that the random variable $F_X(X_{(K)})$ follows a beta law of parameters K and $N-K+1$. The interest of this result is that $F_X(X_{(K)})$ does not depend of X distribution. This allows the derivation of confidence intervals bounding a percentile with a numerical accuracy without knowing neither the values $X_{(i)}$ nor X distribution. For instance, if a DM wants a conservative upper bound of the 95% percentile that covers it with a confidence of at least 95%, then, by using equation (3), we see that at least 59 computations will be required, since with 58 samples, $P(X_{(58)} < X_{95}) = (0.95)^{58} = 5.1\%$ (i.e. a confidence of 94.9 %), while with 59 samples, $P(X_{(59)} < X_{95}) = (0.95)^{59} = 4.8\%$.

In the above case, numerical accuracy has to be expressed as a confidence to cover the true value with the estimation evaluated from the samples. In other situations, the above results cannot be used and numerical accuracy cannot be evaluated beforehand: for example, the DM expresses the desired numerical accuracy as the minimal width of a confidence interval bounding a statistical quantity. This statistical quantity can be a percentile, but also, for example, the mean value (in this case, MC methods converge towards the true value at rate $\sigma/\sqrt{N}$, where σ is the standard deviation).

Since in the RaFu methods, each propagated sample results in an interval with lower and upper bounds, numerical accuracy and confidence intervals have to be given for both of them. After having integrated numerical accuracy in the process by the means of γ_A, we thus end up with two confidence intervals bounding a lower and an upper estimation of the statistical quantity defined by γ_{S_o}.

Figure 7 illustrates the whole procedure by a flowchart. It shows where the DM can act upon the values of parameters and fix them in function of the final desired result. Propagation is then done accordingly, with the minimal number of samples meeting DM requirements. As said before, we focus here on the case where numerical accuracy or number of samples can be determined beforehand (i.e. Yes path in the first diamond).

In order to illustrate our methodology, we provide in Table 1 the minimal sample size given by the RaFu method for various choices of $(\gamma_S, \gamma_E, \gamma_A)$. As stated previously, we focus on percentiles, since percentile is the most relevant statistical quantity in many safety studies. The minimal sample size is derived thanks to the use of order statistics. For example, if the DM wants to have an upper limit of the response 95% percentile assuming stochastic independence between the k random variables, to be hyper-cautious about epistemic uncertainty (i.e. concentrate on α-cuts $[\underline{x}_0, \overline{x}_0]$) and to have a numerical certainty of 99% to cover the true value, he or she chooses the triplet $(\gamma_S, \gamma_E, \gamma_A) = (0.95/\text{Stochastic independence}, 0, 0.99)$. The RaFu method derives the minimal sampling size to satisfy the DM's choice, here 90, and the nature of this sampling. Eventually, if 90 calculations are too costly, the DM can choose to lower the numerical accuracy to 95 %, thus reducing the number of required computations to 59. These two examples are in bold in Table 1.

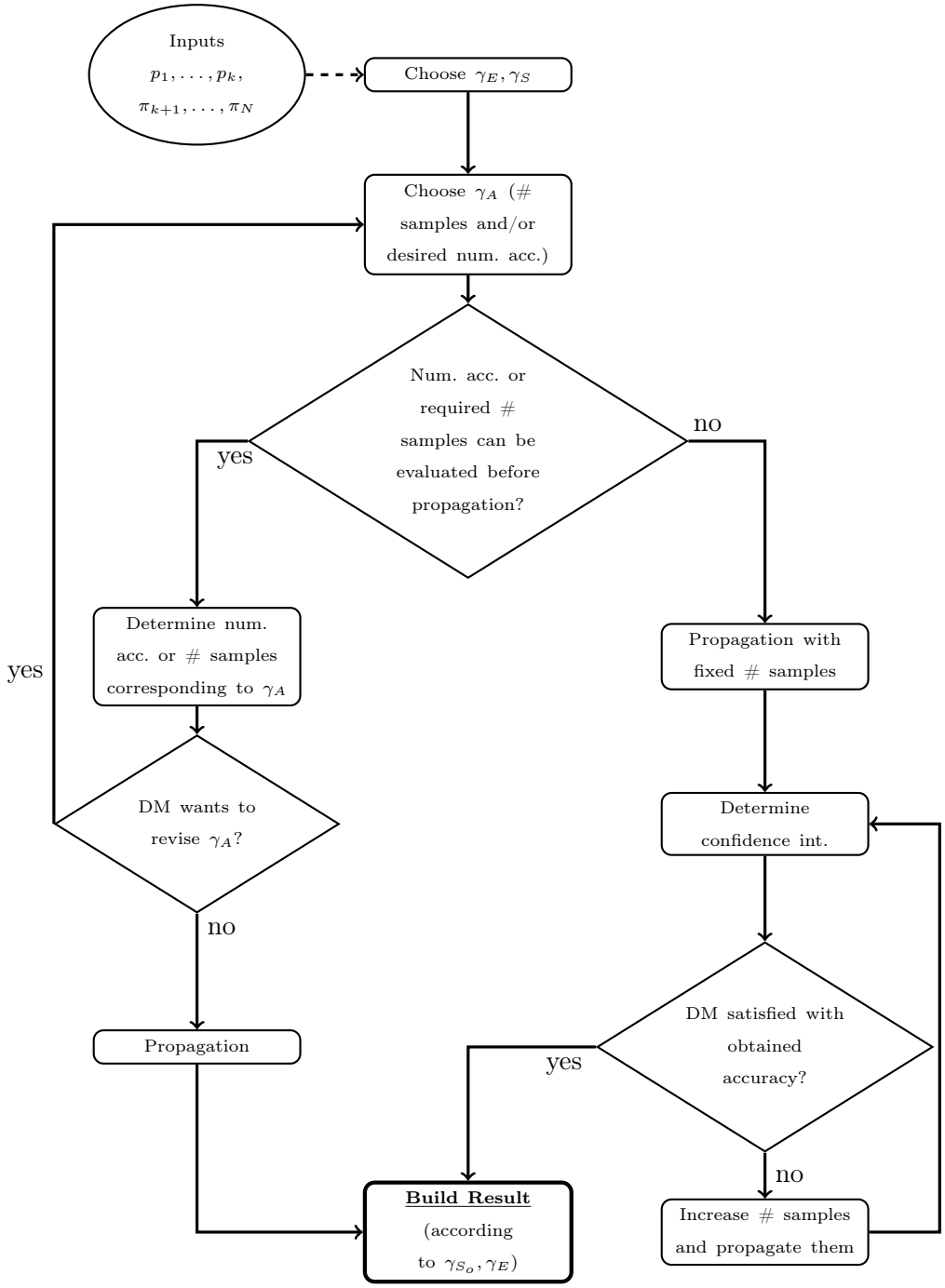

Figure 7. RaFu method : flowchart (# samples: number of samples).

4.2 *Relations with previous post-treatments*

It is interesting to note that the results of post-processing methods in (Ferson and Ginzburg 1996, Baudrit *et al.* 2006) recalled in Section 3, since they are equivalent to evaluate particular integrals over $p_1, \ldots, p_k$, $\pi_{k+1}, \ldots, \pi_N$, can be reached by specific instances of parameter γ_E. We begin by the post-treatment proposed by Baudrit *et al.*.

Proposition 4.1: *The result of the post-treatment giving* $[\underline{F}_{av}, \overline{F}_{av}]$ *can be interpreted as the following choices over* γ_S, γ_E*:*

γ_S	γ_E	# samples		
		$\gamma_A = 90\%$	$\gamma_A = 95\%$	$\gamma_A = 99\%$
90%	α	22	29	44
Stochastic	$(\alpha^1, ..., \alpha^{M_\pi})$	22 $\times M_\pi$	29 $\times M_\pi$	44 $\times M_\pi$
independence	Randomized α for each sample	22	29	44
	Randomized α for each α-cut	22	29	44
95%	α	45	**59**	**90**
Stochastic	$(\alpha^1, ..., \alpha^{M_\pi})$	45 $\times M_\pi$	59 $\times M_\pi$	90 $\times M_\pi$
independence	Randomized α for each sample	45	59	90
	Randomized α for each α-cut	45	59	90
99%	α	230	299	459
Stochastic	$(\alpha^1, ..., \alpha^{M_\pi})$	230 $\times M_\pi$	299 $\times M_\pi$	459 $\times M_\pi$
independence	Randomized α for each sample	230	299	459
	Randomized α for each α-cut	230	299	459

Table 1. Minimal sample size derived by the RaFu method for various choices of $(\gamma_S, \gamma_E, \gamma_A)$. The statistical quantity γ_S is a percentile.

- $\gamma_{S_0} = F(x)\ , \forall x$ *(whole cumulative distribution) and* γ_{S_i} = *Stochastic independence between* $X_1,\ ...,\ X_k$.
- γ_E = *randomized* α *for each sample.*

Proof: Let us consider the model T and the lower probability on Y, $\underline{P}([-\infty, y]) = \underline{F}_Y(y)$, associated to Baudrit *et al.*'s post-treatment. This lower probability corresponds to the lower expectation (also called lower prevision in (Walley 1991)) of the indicator function of the event $[-\infty, y]$. This lower expectation is given by the following formula:

$$\underline{P}([-\infty,y]) = \int_{\kappa=0}^{1} \int_{\alpha_1=0}^{1} \dots \int_{\alpha_k=0}^{1} I(T(F_{X_1}^{-1}(\alpha_1),\dots,F_{X_k}^{-1}(\alpha_k),[\underline{x}_\kappa,\overline{x}_\kappa]_{k+1},\dots,[\underline{x}_\kappa,\overline{x}_\kappa]_N) \subset [-\infty,y]) d\kappa d\alpha_1 \dots d\alpha_k \tag{4}$$

where distributions $P_1, \dots, P_k$ are independent (cf. γ_{S_i}) and $I(A)$ is the indicator function of the event A. Note that the integration of eventual dependencies mentioned in Section 4.1.1 can be easily done and do not modify the present result.

Performing a Monte-Carlo sampling with parameters $(\gamma_S = F(x)\ , \forall x)$ and $(\gamma_E =$ randomized $\alpha)$ for each sample, propagating and then computing the associated lower probability $\underline{P}([-\infty, y])$ is obviously equivalent to a numerical evaluation of the integral given by equation (4). As both Baudrit *et al.*'s approach and the RaFu method are discretized numerical evaluation of the same integral, they converge to the same value.

Since this holds for all values $y \in Y$, the two resulting p-boxes will converge to the same p-box, thus showing that the two methods converge towards the same final result.

For the upper probability on Y, $\overline{P}([-\infty, y]) = \overline{F}_Y(y)$, associated to Baudrit *et al.*'s post-treatment, the reasoning is similar, except that equation (4) becomes

$$\overline{P}([-\infty,y]) = \int_{\kappa=0}^{1} \int_{\alpha_1=0}^{1} \dots \int_{\alpha_k=0}^{1} I(T(F_{X_1}^{-1}(\alpha_1),\dots,F_{X_k}^{-1}(\alpha_k),[\underline{x}_\kappa,\overline{x}_\kappa]_{k+1},\dots,[\underline{x}_\kappa,\overline{x}_\kappa]_N) \cap [-\infty,y] \neq \emptyset) d\kappa d\alpha_1 \dots d\alpha_k \tag{5}$$

This ends the proof. □

We now consider the post-treatment proposed by Ferson. We will consider that

a single value α has been chosen (extension to any number of different values for α is straightforward)

Proposition 4.2: *The result of the post-treatment giving* $[\underline{F}_\kappa, \overline{F}_\kappa]$ *can be interpreted as the following choices over* γ_S, γ_E*:*

- $\gamma_S = F(x)$, $\forall x$ *(whole cumulative distribution) and* γ_{S_i} = *Stochastic independence between* X_1, ..., X_k.
- $\gamma_E = \kappa$.

Proof:

We can use a reasoning similar to the one used in the previous proof, except that now the integral becomes

$$\underline{P}([-\infty,y]) = \int_{\alpha_1=0}^{1} \dots \int_{\alpha_k=0}^{1} I(T(F_{X_1}^{-1}(\alpha_1),\dots,F_{X_k}^{-1}(\alpha_k),[\underline{x}_\kappa,\overline{x}_\kappa]_{k+1},\dots,[\underline{x}_\kappa,\overline{x}_\kappa]_N) \subset [-\infty,y]) d\alpha_1 \dots d\alpha_k \quad (6)$$

and, in particular,

$$\underline{P}([-\infty,y]) = \int_{\alpha_1=0}^{1} \dots \int_{\alpha_k=0}^{1} I(T(F_{X_1}^{-1}(\alpha_1),\dots,F_{X_k}^{-1}(\alpha_k),[\underline{x}_0,\overline{x}_0]_{k+1},\dots,[\underline{x}_0,\overline{x}_0]_N) \subset [-\infty,y]) d\alpha_1 \dots d\alpha_k \quad (7)$$

for $[\underline{F}_0, \overline{F}_0]$, and

$$\underline{P}([-\infty,y]) = \int_{\alpha_1=0}^{1} \dots \int_{\alpha_k=0}^{1} I(T(F_{X_1}^{-1}(\alpha_1),\dots,F_{X_k}^{-1}(\alpha_k),[\underline{x}_1,\overline{x}_1]_{k+1},\dots,[\underline{x}_1,\overline{x}_1]_N) \subset [-\infty,y]) d\alpha_1 \dots d\alpha_k \quad (8)$$

for $[\underline{F}_1, \overline{F}_1]$. □

Even if the RaFu method treats aleatory and epistemic uncertainty with the same theoretical framework as Baudrit *et al.*'s and Ferson's approaches, the required number of samples (i.e. of computations) leading to the same results can be very different. Table 2 compares the numerical requirements of the various approaches for the particular example given at the end of Section 3, in order to compute the resulting p-boxes of each post-processing. This table illustrates the main advantage of the proposed method versus usual post-processings: since it concentrates exclusively on the desired final answer, it only propagates the core information needed to reach this answer. It allows, in this situation, to divide the number of required propagation by 20 (resp. 10) compared to Baudrit *et al.* (resp. Ferson) or by a number proportional to M_π in a more general case. From a computational efficiency standpoint, this is an important improvement keeping in mind that industrial applications involve complex computer codes.

Post-processing (with fixed γ_S, γ_A)	Usual propagation (build the whole RFV)	RaFu Method	
Baudrit *et al*	# samples: 2100 $M_p = 100, M_\pi = 21$	γ_E (Strat 3)	# samples: 100 $M_p = 100$
Ferson	# samples: 2100 $M_p = 100, M_\pi = 21$	γ_E (Strat. 2) $= \{\alpha = (0,1)\}$	# samples: 200 $M_p = 100, M_\pi = 2$

Table 2. Comparison between classical post-processings and the RaFu method.

Symbol	Name
Q	River flow rate
B	River width
K_s	Strickler coefficient
Z_u	Upriver water level
Z_d	Downriver water level
L	River length

Table 3. Summary of parameters used in equation (9)

5. Illustration

In this section, we apply the RaFu method on a simplified model used by EDF (French integrated energy operator) to compute the overflowing height for a river dike (Magne and Vasseur 2006). Although this model is quite simple, it provides a realistic industrial application in which we can distinguish between aleatory and epistemic uncertainty. This model approximates the overflowing height H of a river and depends on six parameters which are summarized in Table 3. It reads

$$H = \left(\frac{Q}{K_s \sqrt{\frac{Z_u - Z_d}{L}} B} \right)^{\frac{3}{5}} \tag{9}$$

5.1 *Modeling uncertainty sources*

We assume the river width (B) is constant on all the length of the river (L). Both these width and length are assumed to be well known (i.e. no uncertainty on these parameters). They are set to $L = 5000m$ and $B = 300m$ in the following numerical tests.

The value of the river flow rate (Q) depends on a huge number of physical phenomena (e.g. climatic and meteorologic conditions, period of the year, ...) that are highly variable over time and/or space. The flow rate value can therefore be interpreted as an aleatory value due to the natural variability of various physical phenomena. As a lot of measurements are usually available for river flow rates, it is possible to fit the data to a probability law modeling this variability. Experience has shown that this variability can be well represented by classical lognormal or Gumbel laws.

Water levels Z_u and Z_d depend on sedimentary conditions that are peculiar to the considered river bed. Due to various reasons, these sedimentary conditions are usually not well known, but are not the consequence of some physical variability or of some random event (since we consider a specific river). The uncertainty of the water levels being due to a lack of information, it is therefore of epistemic nature, and should be modeled by a fuzzy variable.

Similarly, the Strickler coefficient K_s is a model parameter used instead of a physical model to describe the dependance between the flow velocity and the slope of the river. It is also specific to the considered river bed, and the complexity of the river nature makes it difficult to estimate with precision. In our context, the uncertainty linked to such a non-measurable parameter should be modeled by a fuzzy variable as well. Table 4 gives the chosen models for our application (considered values and uncertainties are typical values). As an example, Figure 8 illustrates the possibility distribution modeling the epistemic uncertainty on K_s.

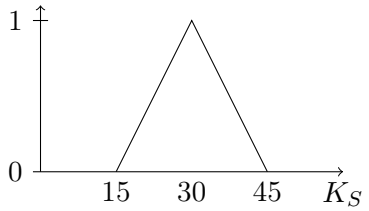

Figure 8. Triangular fuzzy number modeling K_s

Variable	Unit	Model
Q	m^3s^{-1}	Lognormal law ($m = 7.04$ and $\sigma = 0.6$)
Z_u	m	Triangular possibility distribution (54,55,56)
Z_d	m	Triangular possibility distribution (49,50,51)
K_s	$m^{1/3}s^{-1}$	Triangular possibility distribution (15,30,45)

Table 4. Uncertainty models without the RaFu approach

For sake of completeness, we also consider in the following applications the results derived from a probabilistic approach. In this case,uncertainties are modeled by the means of probability distributions. According to the available knowledge on uncertain parameters, we therefore associate to Z_u, Z_d and K_S triangular probability distributions with the same support and mode as the triangular fuzzy numbers of Table 4. Even if Figure 8 can be interpreted, up to normalization, as the triangular probability distribution modeling the aleatory uncertainty attached to K_S, there is however a fundamental difference related to the information represented by these two types of distributions. As mentionned in Section 2.1.2, a possibility distribution can be seen as a partial probabilistic model. The triangular possibility distribution $(15, 30, 45)$ is then more similar to a set of probability distributions characterized by "the most likely value is 30" and "the support is [15;45]" (i.e. Z_S cannot take any values outside $[15; 45]$) whereas the triangular probability distribution $(15, 30, 45)$ is just a specific probability distribution in this set. We refer to (Dubois and Prade 1992, de Cooman and Aeyels 1999) for more details.

5.2 *Numerical tests*

To illustrate the RaFu approach, we give various results where the RaFu method is applied with a fixed number of samples ($M = 1000$). We first illustrate the construction of whole cumulative distributions corresponding to the two post-processings of Baudrit *et al.* and Ferson. We then shows the effect of using numerical approximation on the the 95% percentile.

In these applications, γ_S is the whole cumulative distribution (resp. the 95% percentile) with stochastic independence whereas γ_E corresponds to Ferson's post-treatment (i.e.$\alpha = \{0, 1\}$ or Strategy 2) and to Baudrit *et al.*'s post-treatment ($\alpha = \{av\}$ or Strategy 3).

5.2.1 Cumulative distributions

Figure 9 displays the CDF(s) associated to the RaFu method and to the probabilistic approach. For sake of clarity, numerical accuracy (i.e. γ_A) is not considered in this figure.

Note that, because fuzzy variables (epistemic uncertainty) are modeled by means of triangular fuzzy numbers, taking $\gamma_E = \{\alpha = 1\}$ comes down to suppress this epis-

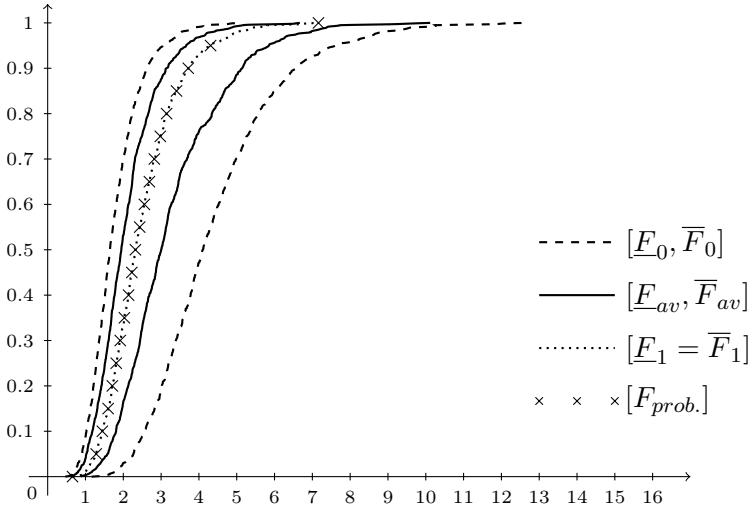

Figure 9. Result of Rafu Method and probabilistic approach with 1000 samples

temic uncertainty, thus the result is a unique cumulative distribution (we consider that both Z_u, Z_d, K_s are precisely known).

It is important to notice that the pair(s) of CDFs corresponding to Ferson's and Baudrit et al.'s post-treatments have been derived by the RaFu method using 1000 samples. Had we built the whole random fuzzy variable which is required in the two previous post-treatments, $M_\pi \times 1000$ interval computations would have been necessary, with M_π the chosen number of discretized α-cuts.

Note that the CDF of the probabilistic approach $F_{prob.}$ is encompassed by the P-boxes $[\underline{F}_0, \overline{F}_0]$ and $[\underline{F}_{av}, \overline{F}_{av}]$. It is also quite similar to the cumulative distribution $\underline{F}_1$. Therefore, uncertainty on Z_u, Z_d and K_S have almost no effects on the result with a fully probabilistic approach, while it is not the case with an hybrid approach explicitly differentiating epistemic and aleatory uncertainty. This well shows that choosing an uncertainty model does have an important effect, even for simple examples.

5.2.2 Numerical accuracy of 95% *percentile*

Figure 10 illustrates numerical accuracy on the estimation of the 95% percentile. For each case considered in Figure 9, it displays the best estimates of the 95% percentiles, as well as lower and upper bounds bounding these percentiles with a 95% confidence (i.e. $\gamma_A = 95\%$). Since for $\gamma_E = \{\alpha = 0\}$ and the probabilistic approach, intervals reduce to single values, we have six series of 1000 values (corresponding to lower/upper bounds of $\gamma_E = \{\alpha = 1, 0, av\}$ and to the probabilistic approach). From these samples, the 95% percentile and its numerical accuracy are then estimated using order statistics and Equation (3). More precisely, the lower and upper bounds of the 95 % numerical confidence interval respectively correspond to the 936^{th} and 964^{th} sorted values. Best estimates are given by the 950^{th} sorted value.

It comes out that the introduction of imprecision in the uncertainty analysis and the DM's behavior with respect to epistemic uncertainty can strongly affect the estimation of the 95% percentile. For example, for a hyper-cautious DM, a difference of $4.5m$ is noticeable whereas an optimistic strategy provides roughly the same estimation as in the probabilistic case. Moreover, the numerical approximation effect is not negligible, even for a relatively high number of computations (here, 1000). This last result is all the more important as, in pratical safety studies dealing with complex computer codes, the number of affordable computations does not exceed

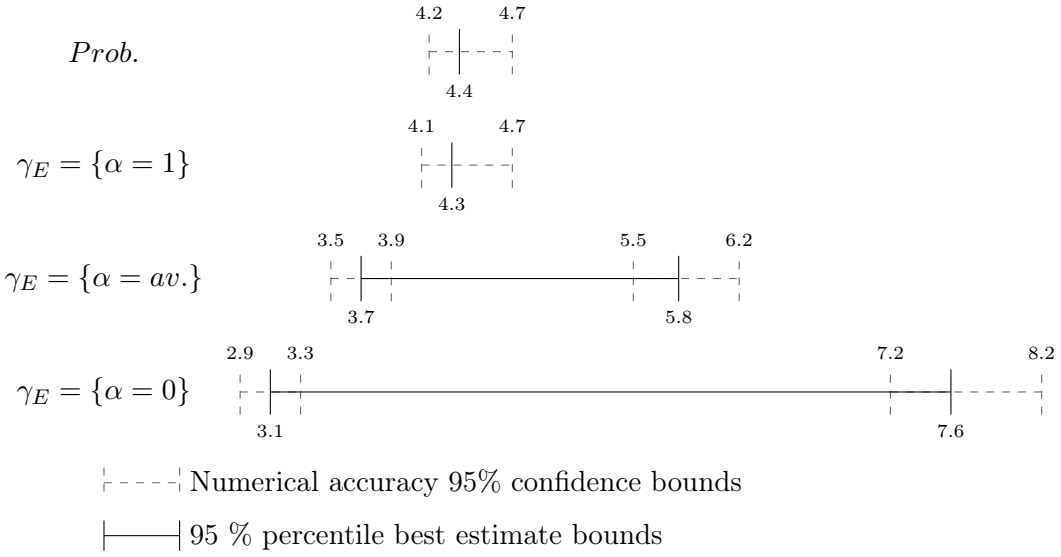

Figure 10. Evaluation of the 95% percentile

200 (OCDE 2006, Baccou and Chojnacki 2007). Therefore, the numerical uncertainty margin can be even larger than in our example. Imprecision and numerical error approximation can have potential consequence on the decision-making process (compare the $4.4m$ of the probabilistic approach to the $8.2m$ of the RaFu method in the case of a hyper-cautious DM with maximal numerical error).

6. Conclusions and perspectives

Mixing fuzzy and probabilistic modeling allows one to differentiate epistemic and aleatory uncertainties, and to take account of both of them. After propagation of probabilities and fuzzy sets by their respective calculus, one obtains a random fuzzy random, to which is usually applied a post-processing allowing to get a summarized information. Two shortcomings of this procedures are the followings: first, it has an important computational cost which prevents the use of such methods to complex models; second, while using a numerical approximation in the propagation, it does not consider the issue of evaluating the numerical error arising from such approximations.

We have introduced a new numerical method (the RaFu method) that addresses these two issues in the following ways:

- it reduces the computational burden of the propagation by concentrating on the final desired result, only propagating the minimal information to obtain it. In other words, it avoids building the whole random fuzzy variable when unnecessary.
- It evaluates the numerical error resulting from the approximation due to sampling strategies, by using features of M-C simulations techniques and order statistics.

In this method, the decision maker (DM) specifies before the uncertainty propagation which information he's interested in, and what is the numerical accuracy he wants to reach. An optimized sampling strategy is then determined, which minimizes the number of required computations.

Since epistemic and aleatory uncertainties are often mixed (i.e. imprecisely observed random variables), the next step would be to integrate imprecise probabilities

into the picture: for example by considering and sampling from imprecisely defined cumulative distributions (so-called p-boxes (Ferson *et al.* 2003)) instead of precise ones. In this perspective, IRSN is currently studying how usual numerical methods designed to handle dependencies (Iman and Conover 1982) can be extended to such models (Destercke and Chojnacki 2008). Also, it would be interesting to study how recent sampling techniques such as MCMC methods could be applied to the present framework, in order to improve numerical accuracy and computational efficiency even more.

Another issue is to consider epistemic uncertainty concerning second order models: in applications, it often happens that one knows the type of parametric law modeling a population (e.g. gaussian, exponential law, law corresponding to extreme values) but only knows the parameters defining this law with imprecision. It would be interesting to model this epistemic uncertainty by means of possibility distributions, and to study the effect of various assumptions on the propagation of such an uncertainty. For example, in the application concerning the overflowing height of a river, it can happen that the parameters of the lognormal law modeling the river flow rates are only imprecisely known (due to systematic errors in measurements or to the limited number of measurements made on a specific river).

Finally, we would like to point out that the current study settles in a much more general problematic, which is the search of efficient simulation techniques and efficient computational methods allowing to deal with imprecise probabilities.

References

Alvarez, D.A., 2006. On the calculation of the bounds of probability of events using infinite random sets. *Int. J. of Approximate Reasoning*, 43, 241–267.

Baccou, J. and Chojnacki, E., 2007. Contribution of the mathematical modelling of knowledge to the evaluation of uncertainty margins of a LBLOCA transient (LOFT-L2-5). *Nuclear Engineering and Design*, 237, 2064–2074.

Bardossy, G. and Fodor, J., 2004. *Evaluation of uncertainties and risks in geology: new mathematical approaches for their handling*. Berlin: Springer.

Baudrit, C., Dubois, D. and Perrot, N., 2008. Representing parametric probabilistic models tainted with imprecision. *Fuzzy Sets and Systems*, 159, 1913–1928.

Baudrit, C., Guyonnet, D. and Dubois, D., 2006. Joint Propagation and Exploitation of Probabilistic and Possibilistic Information in Risk Assessment. *IEEE Trans. Fuzzy Systems*, 14, 593–608.

Birnbaum, A., 1961. Confidence curves: an omnibus technique for estimation and testing statistical hypothesis. *Journal of American Statistical Association*, 56, 246–249.

Conover, W., 1999. *Practical non-parametric statistic*. 3rd New York: Wiley.

Cooper, J., Ferson, S. and Ginzburg, L., 1996. Hybrid processing of stochastic and subjective uncertainty. *Risk Analysis*, 16, 785–791.

Couso, I., Moral, S. and Walley, P., 2000. A survey of concepts of independence for imprecise probabilities. *Risk Decision and Policy*, 5, 165–181.

Cox, D., 1958. Some problems connected with statistical inference. *Annals of Mathematical Statistics*, 29, 357–372.

de Cooman, G. and Aeyels, D., 1999. Supremum-preserving upper probabilities. *Information Sciences*, 118, 173–212.

Destercke, S. and Chojnacki, E., 2008. Extending usual methods handling dependencies to imprecise probabilistic models. .

Dubois, D., Nguyen, H. and Prade, H., 2000. Possibility theory, probability and

fuzzy sets: misunderstandings, bridges and gaps. *In*: *Fundamentals of Fuzzy Sets.*, 343–438 D. Dubois and H. Prade.

Dubois, D. and Prade, H., 1988. *Possibility Theory: An Approach to Computerized Processing of Uncertainty*. New York: Plenum Press.

Dubois, D. and Prade, H., 1992. When upper probabilities are possibility measures. *Fuzzy Sets and Systems*, 49, 65–74.

Feller, W., 1971. *An Introduction to Probability Theory and Its Applications*. Vol. I & II. New York: John Wiley and Sons.

Ferson, S., Ginzburg, L., Kreinovich, V., Myers, D. and Sentz, K., Constructing probability boxes and Dempster-Shafer structures. , 2003. , Technical report, Sandia National Laboratories.

Ferson, S. and Ginzburg, L.R., 1996. Different methods are needed to propagate ignorance and variability. *Reliability Engineering and System Safety*, 54, 133–144.

Helton, J. and Oberkampf, W., eds. , 2004. *Alternative Representations of Uncertainty, Special issue of Reliability Engineering and Systems Safety*. Vol. 85. Elsevier.

Iman, R. and Conover, W., 1982. A distribution-free approach to inducing rank correlation among input variables. *Communications in Statistics*, 11 (3), 311–334.

Jaulin, L., Kieffer, M., Didrit, O. and Walter, E., 2001. *Applied Interval Analysis*. London.

Kaufmann, A. and Gupta, M., 1985. *Introduction to Fuzzy Arithmetic: Theory and Applications*. .

Lecoutre, J. and Tassi, P., 1987. *Statistique non-parametrique et robustesse*. Economica.

Magne, L. and Vasseur, D., eds., 2006. Maitriser les incertitudes pour mieux gerer les risques. *In*: *Risques industriels. Complexite, incertitude et decision : une approche interdisciplinaire.*, 219–260 Lavoisier.

Moore, R., 1979. *Methods and applications of Interval Analysis*. SIAM Studies in Applied Mathematics Philadelphia: SIAM.

Nelsen, R., 2005. Copulas and quasi-copulas: An introduction to their properties and applications. *In*: E. Klement and R. Mesiar, eds. *Logical, Algebraic, Analytic, and Probabilistics Aspects of Triangular Norms*. Elsevier, chap. 14.

Oberguggenberger, M., King, J. and Schmelzer, B., 2007. Imprecise probability methods for sensitivity analysis in engineering. 317–326.

OCDE, Bemuse Phase 1,2,3 Report. , 2006. , Technical report NEA/CSNI/R(2006), Nuclear Energy Agency (NEA).

Regan, H., Ferson, S. and Berleant, D., 2004. Equivalence of methods for uncertainty propagation of real-valued random variables. *Int. J. of Approximate Reasoning*, 36, 1–30.

Walley, P., 1991. *Statistical reasoning with imprecise Probabilities*. New York: Chapman and Hall.

Wilks, S., 1962. *Mathematical statistics*. New York.