Generalised p-boxes on totally ordered spaces

E. Miranda[1], M. C. M. Troffaes[2], and S. Destercke[3]

Abstract Probability boxes are among the most simple and popular models used in imprecise probability theory, and many practical results concerning them exist in the literature. Nevertheless, little attention has been paid to their formal characterisation in the setting of Walley's behavioural theory of imprecise probabilities. This paper tries to remedy this situation by formalising, generalising and extending existing results as well as by giving new ones, within Walley's framework.

Key words: Probability boxes, ordered spaces, limit representation, characterization

1 Introduction

Imprecise probability [7] is a generic term referring to uncertainty models where the available information does not allow singling out a unique probability measure. Unlike classical probability models, which are uniquely determined by their values on events, general imprecise probability models are determined by bounds on expectations of random variables [7, p. 82, 2.7.3]. This more advanced mathematical description allows more flexibility in the representation, but also implies more complexity when treating uncertainty.

For this reason, it is of interest to consider particular imprecise probability models that yield simpler mathematical descriptions, at the expense of generality, but

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Enrique Miranda
Rey Juan Carlos University, Dept. of Statistics and O.R. C-Tulipán, s/n, Móstoles, Spain `enrique.miranda@urjc.es`

Matthias Troffaes
Durham University, Department of Mathematical Sciences, Science Laboratories, South Road, Durham DH1 3LE, United Kingdom `matthias.troffaes@gmail.com`

Sébastien Destercke
Institut de Radioprotection et de Sureté Nucléaire, Bat 702, 13115 St-Paul lez Durance, Cadarache, France `sebastien.destercke@irsn.fr`

gaining ease of use, elicitation, and graphical representation. One of such models is considered in this paper: pairs of lower and upper cumulative distribution functions, also called *probability boxes*, or briefly, p-boxes [3]. Practical aspects of this model have been extensively studied in the literature, but little attention has been given to their formal characterisation in terms of lower and upper expectations, or, equivalently, of coherent lower previsions (they are briefly studied in [6, 7], and in [4] cumulative distribution functions associated with a sequence of moments are considered).

This paper aims at such study, and considers a generalised version of p-boxes, defined on any (not necessarily finite) totally ordered space. In [2], a similar extension on total pre-ordered finite spaces is considered. This paper formulation covers generalised p-boxes defined on totally ordered finite spaces as well as on closed real intervals. More generally, such treatment also admits p-boxes on product spaces (by considering an appropriate order), and thus admits imprecise multivariate distributions through p-boxes as well.

The paper is organised as follows: Section 2 provides a brief introduction to the theory of coherent lower previsions. Section 3 then introduces and studies the p-box model from the point of view of lower previsions. Section 4 provides a first expression for the natural extension of a p-box, and studies its main properties. In Section 5 we prove that any p-box can be approximated as a limit of discrete p-boxes, and that this limit holds into the natural extensions. Finally, we end in Section 6 with main conclusions and open problems. Due to limitations of space, proofs have been omitted.

2 Preliminaries

Let us briefly introduce coherent lower previsions; see [7] for more details. Let Ω be the possibility space. A subset of Ω is called an *event*. A *gamble* on Ω is a bounded real-valued function on Ω. The set of all gambles on Ω is denoted by $\mathscr{L}(\Omega)$, or simply by $\mathscr{L}$ if the possibility space is clear from the context. A particular type of gamble is the *indicator* of an event A, which is the gamble that takes the value 1 on elements of A and the value 0 elsewhere, and is denoted by I_A, or simply by A if no confusion is possible.

A *lower prevision* $\underline{P}$ is a real-valued functional defined on an arbitrary subset $\mathscr{K}$ of $\mathscr{L}$. If f is a gamble, $\underline{P}(f)$ is interpreted as the maximum buying price for the (uncertain) reward f. It can be argued that lower previsions model a subject's belief about the true state x in Ω. A lower prevision defined on a set of indicators of events is usually called a *lower probability*.

A lower prevision on $\mathscr{K}$ is called *coherent* when for all p in $\mathbb{N}$, all $f_0, f_1, \dots, f_p$ in $\mathscr{K}$ and all $\lambda_0, \lambda_1, \dots, \lambda_p$ in $\mathbb{R}^+$,

$$\sup_{x\in\Omega}\left[\sum_{i=1}^{p}\lambda_i(f_i-\underline{P}(f_i))-\lambda_0(f_0-\underline{P}(f_0))(x)\right]\geq 0.$$

A lower prevision on the set $\mathscr{L}$ of all gambles is coherent if and only if

(C1) $\underline{P}(f)\geq\inf f$,
(C2) $\underline{P}(\lambda f)=\lambda\underline{P}(f)$, and

(C3) $\underline{P}(f+g) \geq \underline{P}(f)+\underline{P}(g)$
for all gambles f, g and all non-negative real numbers λ. A lower prevision on $\mathscr{L}$ satisfying (C3) with equality for all gambles f and g is called a *linear prevision* on $\mathscr{L}$, and the set of all linear previsions on $\mathscr{L}$ is denoted by $\mathscr{P}$. A lower prevision $\underline{P}$ on $\mathscr{K}$ can also be characterised by the set

$$\mathscr{M}(\underline{P}) = \{Q \in \mathscr{P} \colon (\forall f \in \mathscr{K})(Q(f) \geq \underline{P}(f))\}.$$

Then $\underline{P}$ is coherent if and only if $\underline{P}(f) = \min_{Q\in\mathscr{M}(\underline{P})} Q(f)$ for all $f \in \mathscr{K}$.

Given a coherent lower prevision $\underline{P}$ on $\mathscr{K}$, its *natural extension* to a larger set $\mathscr{K}_1 \supseteq \mathscr{K}$ is the pointwise smallest coherent (i.e., least-committal) lower prevision on $\mathscr{K}_1$ that agrees with $\underline{P}$ on $\mathscr{K}$. The procedure of natural extension is transitive [6, p. 98]: if $\underline{E}_1$ is the natural extension of $\underline{P}$ to $\mathscr{K}_1$ and $\underline{E}_2$ is the natural extension of $\underline{E}_1$ to $\mathscr{K}_2 \supseteq \mathscr{K}_1$, then $\underline{E}_2$ is also the natural extension of $\underline{P}$ to $\mathscr{K}_2$. The natural extension to all gambles is usually denoted by $\underline{E}$. It holds that $\underline{E}(f) = \min_{Q\in\mathscr{M}(\underline{P})} Q(f)$ for any $f \in \mathscr{L}$.

A particular class of coherent lower previsions of interest in this paper are *completely monotone lower previsions* [1]. A lower prevision $\underline{P}$ defined on a lattice of gambles $\mathscr{K}$ is called n-monotone when for all $p \in \mathbb{N}$, $p \leq n$, and all f, $f_1, \dots, f_p$ in $\mathscr{K}$:

$$\textstyle\sum_{I\subseteq\{1,\dots,p\}}(-1)^{|I|}\underline{P}(f \wedge \bigwedge_{i\in I} f_i) \geq 0,$$

and is called *completely monotone* when it is n-monotone for all $n \in \mathbb{N}$.

3 Characterising p-boxes

Let $(\Omega, \leq)$ be an order complete chain. Let $x < y$ be a brief notation for $x \leq y$ and $x \ngeq y$. So $\leq$ is transitive, reflexive, and anti-symmetric, and for any two elements x, $y \in \Omega$ we have either $x < y$, $x = y$, or $x > y$. For simplicity, we assume that Ω has a smallest element 0_Ω and a largest element 1_Ω.

We call *cumulative distribution function* any non-decreasing function $F : \Omega \to [0,1]$ that satisfies $F(1_\Omega) = 1$. $F(x)$ provides information about the cumulative probability on the interval $[0_\Omega, x]$. Note that we do not need to impose $F(0_\Omega) = 0$. Also note that cumulative distribution functions are not assumed to be right-continuous. Given a cumulative distribution F on Ω and a value $x \in \Omega$, $F(x^+)$ is the right-limit and $F(x^-)$ is the left-limit,

$$F(x^+) = \inf_{y>x} F(y) = \lim_{y\to x,\, y>x} F(y) \qquad F(x^-) = \sup_{y<x} F(y) = \lim_{y\to x,\, y<x} F(y)$$

and $F(1_\Omega^+) = 1$ and $F(0_\Omega^-) = 0$.

Definition 1. A *generalised probability box*, or *generalised p-box*, is a pair $(\underline{F}, \overline{F})$ of cumulative distribution functions from Ω to $[0,1]$, satisfying $\underline{F} \leq \overline{F}$. If Ω is a closed interval on $\mathbb{R}$, then we call the pair $(\underline{F}, \overline{F})$ a *p-box*.

A generalised p-box is interpreted as a lower and an upper cumulative distribution function. In Walley's framework, this means that a generalised p-box is interpreted as a lower prevision (actually a lower probability) $\underline{P}_{\underline{F},\overline{F}}$ on the set of events

$$\mathscr{K} = \{[0_\Omega, x] \colon x \in \Omega\} \cup \{(y, 1_\Omega] \colon y \in \Omega\}$$

by

$$\underline{P}_{\underline{F},\overline{F}}([0_\Omega, x]) := \underline{F}(x) \text{ and } \underline{P}_{\underline{F},\overline{F}}((y, 1_\Omega]) = 1 - \overline{F}(y).$$

In the particular case of p-boxes it was mentioned by [7, Section 4.6.6] and proven by [6, p. 93] that $\underline{P}_{\underline{F},\overline{F}}$ is coherent. It is straightforward to show that generalised p-boxes are coherent as well.

Given a generalised p-box, we can consider the set of cumulative distribution functions that lie between $\underline{F}$ and $\overline{F}$,

$$\Phi(\underline{F}, \overline{F}) = \left\{F \colon \underline{F} \leq F \leq \overline{F}\right\}.$$

We can easily express the natural extension $\underline{E}_{\underline{F},\overline{F}}$ in terms of $\Phi(\underline{F}, \overline{F})$: $\underline{E}_{\underline{F},\overline{F}}$ is the lower envelope of the natural extensions of the F between $\underline{F}$ and $\overline{F}$:

$$\underline{E}_{\underline{F},\overline{F}}(f) = \inf_{F \in \Phi(\underline{F},\overline{F})} \underline{E}_F(f) \tag{1}$$

for all gambles f on Ω. A similar result for p-boxes in the unit interval can be found in [7, Section 4.6.6].

Next, we study the natural extension of a generalised p-box, that is, what information a generalised p-box provides about the buying prices for the gambles which are not in $\mathscr{K}$. For this, we shall regularly invoke the field of events $\mathscr{H}$ generated by the domain $\mathscr{K}$, i.e., events of the type

$$[0_\Omega, x_1] \cup (x_2, x_3] \cup \cdots \cup (x_{2n}, x_{2n+1}]$$

for $x_1 < x_2 < x_3 < \cdots < x_{2n+1}$ in Ω (if n is 0 then this is $[0_\Omega, x_1]$) and

$$(x_2, x_3] \cup \cdots \cup (x_{2n}, x_{2n+1}]$$

for $x_2 < x_3 < \cdots < x_{2n+1}$ in Ω.

Since the procedure of natural extension is transitive, in order to calculate the natural extension of $\underline{P}_{\underline{F},\overline{F}}$ to all gambles we shall first consider the extension from $\mathscr{K}$ to $\mathscr{H}$, then the natural extension from $\mathscr{H}$ to the set of all events, and finally the natural extension from the set of all events to the set of all gambles. The first of these steps is achieved by the following proposition:

Proposition 1. *Given $A = [0_\Omega, x_1] \cup (x_2, x_3] \cup \cdots \cup (x_{2n}, x_{2n+1}]$,*

$$\underline{E}_{\underline{F},\overline{F}}(A) = \underline{F}(x_1) + \sum_{k=1}^{n} \max\{0, \underline{F}(x_{2k+1}) - \overline{F}(x_{2k})\}$$

and given $A = (x_2, x_3] \cup \cdots \cup (x_{2n}, x_{2n+1}]$,

$$\underline{E}_{\underline{F},\overline{F}}(A) = \sum_{k=1}^{n} \max\{0, \underline{F}(x_{2k+1}) - \overline{F}(x_{2k})\}.$$

We now describe the natural extension of a generalised p-box by a Choquet integral.

4 The natural extension as a Choquet integral

As shown in [4, Section 3.1], the natural extension $\underline{E}_F$ of a cumulative distribution function F on $[0,1]$ is completely monotone. It is fairly easy to generalise this result to cumulative distribution functions on a totally ordered space Ω. In this section we establish this for generalised p-boxes.

Theorem 1. *The natural extension $\underline{E}_{\underline{F},\overline{F}}$ of $\underline{P}_{\underline{F},\overline{F}}$ to $\mathscr{L}(\Omega)$ is given by the Choquet integral $(C)\int \cdot \,\mathrm{d}\underline{P}^{\mathscr{H}}_{\underline{F},\overline{F}*}$, where $\underline{P}^{\mathscr{H}}_{\underline{F},\overline{F}*}$ is the inner measure of $\underline{P}^{\mathscr{H}}_{\underline{F},\overline{F}}$,*

$$\underline{P}^{\mathscr{H}}_{\underline{F},\overline{F}*}(A) = \sup_{C \in \mathscr{H}, C \subseteq A} \underline{P}^{\mathscr{H}}_{\underline{F},\overline{F}}(C). \tag{2}$$

Moreover, $\underline{E}_{\underline{F},\overline{F}}$ is a completely monotone lower prevision.

The remainder of this section is devoted to the study of this natural extension, in order to provide more manageable expressions for it. We shall characterise $\underline{E}$ by the values it takes on intervals of the form $[0_\Omega, x], (x,y], [0_\Omega, x)$ and (x,y), for $x \le y$ in Ω, through the lower oscillation of gambles and full components of events, as explained further on. For ease of notation, we shall denote $\underline{E}_{\underline{F},\overline{F}}$ by $\underline{E}$ when no confusion is possible.

Let us consider the *upper limit topology* on Ω which is the topology generated by the base $\tau := \{(x,y] : x,y \in \Omega, x < y\} \cup \{[0_\Omega, x] : x \in \Omega\}$. For any gamble f on Ω, let us define its *lower oscillation* as the gamble

$$\underline{osc}(f)(d) := \sup_{C \in \tau \colon d \in C} \inf_{x \in C} f(x);$$

given $A \subseteq \Omega$, the lower oscillation of I_A is the indicator function of

$$B := \{d \in A : \exists C \in \tau \text{ s.t. } d \in C \subseteq A\} = \bigcup_{C \in \tau \colon C \subseteq A} C = \mathrm{int}(A); \tag{3}$$

note that B is the union of the elements of the base τ that are included in A, and is therefore the topological interior of A in the upper limit topology. It is not too difficult to show that the lower oscillation of f is the supremum of all continuous gambles (with respect to the upper limit topology) that are dominated by f.

Lemma 1. *For any subset A of Ω, $\underline{E}(A) = \underline{E}(B)$, where B is given by Eq.* (3).

This lemma allows us to deduce the following characterisation of $\underline{E}$:

Proposition 2. *For any gamble f on Ω, $\underline{E}(f) = \underline{E}(\underline{osc}(f))$.*

This result allows us to rewrite the Choquet integral of Theorem 1 as

$$\underline{E}(f) = \inf \underline{osc}(f) + \int_{\inf \underline{osc}(f)}^{\sup \underline{osc}(f)} \underline{E}(\{\underline{osc}(f) \geq x\})dx = \underline{E}(\underline{osc}(f)), \tag{4}$$

which is indeed more manageable. Note that for any $t \in \mathbb{R}$, $\{\underline{osc}(f) > t\}$ is equal to $\underline{osc}(\{f > t\})$, and as consequence $\underline{osc}$ is a lower semi-continuous function if we consider the upper limit topology in the initial space. Hence, the natural extension of a generalised p-box is characterised by its restriction to lower semi-continuous gambles (and, because of Eq. (4), to open sets). Taking this into account, we are going to determine the expression of the natural extension $\underline{E}$ on the subsets of Ω which are open in the upper limit topology.

Let B be an open subset of Ω, and let us show that B is a union of pairwise disjoint open intervals of Ω. Recall that by open we are referring here to the upper limit topology, so the subinterval $(a,b]$ is also open for any a,b in Ω.

Definition 2. [5] A set S is called *full* if $[a,b] \subseteq S$ for any $a \leq b$ in S. Given a set A and an element x of A, the *full component* $C(x,A)$ of x in A is the largest full set S which satisfies $x \in S \subseteq A$.

The full components $\{C(x,A)\colon x \in A\}$ of a set $A \subseteq \Omega$ form a partition of A [5, 4.4(a)]. In the following lemma, we prove that the natural extension $\underline{E}$ is additive on full components.

Lemma 2. *Let A be an arbitrary subset of Ω, and let $(A_\lambda)_{\lambda\in\Lambda}$ be the full components of A. Then $\underline{E}(\cup_{\lambda\in\Lambda} A_\lambda) = \sum_{\lambda\in\Lambda} \underline{E}(A_\lambda)$. If moreover A is open, then A_λ is open for all $\lambda \in \Lambda$.*

So the natural extension $\underline{E}$ is characterised by the value it takes on the full components of open sets. By Lemma 2, these full components are open intervals of Ω, and are therefore of the form $[0_\Omega,x],(x,y],[0_\Omega,x)$ or (x,y), for $x \leq y$ in Ω. By Proposition 1 we have that $\underline{E}([0_\Omega,x]) = \underline{F}(x)$ and $\underline{E}((x,y]) = \max\{0,\underline{F}(y) - \overline{F}(x)\}$ for any $y \leq x$ in Ω, and by Eq. (2),

$$\underline{E}([0_\Omega,x)) = \underline{F}(x-) \text{ and } \underline{E}(x,y) = \max\{0,\underline{F}(y-) - \overline{F}(x)\}.$$

5 Limit approximations of the natural extension

Next, we give an alternative expression of the natural extension of a generalised p-box as a limit of the natural extensions of discrete p-boxes. Consider a p-box $(\underline{F},\overline{F})$ on Ω. Let $(\underline{F}_n)_n,(\overline{F}_n)_n$ be increasing and decreasing sequences of cdfs converging point-wise to $\underline{F}$ and $\overline{F}$, respectively.

For ease of notation, denote by $\underline{P}_n$ the lower probability associated with $(\underline{F}_n,\overline{F}_n)$, that is, $\underline{P}_n = \underline{P}^{\mathscr{H}}_{\underline{F}_n,\overline{F}_n}$ and let $\underline{E}_n$ be natural extension of $\underline{P}_n$. Since $\underline{F}_n \leq \underline{F}$ and $\overline{F}_n \geq \overline{F}$, it follows that $\Phi(\underline{F},\overline{F}) \subseteq \Phi(\underline{F}_n,\overline{F}_n)$, and Eq. (1) implies that $\underline{E}_n \leq \underline{E}$. Moreover, the same argument implies that $\underline{E}_n \leq \underline{E}_{n+1}$ for any $n \in \mathbb{N}$, so $\lim_n \underline{E}_n = \sup_n \underline{E}_n \leq \underline{E}$. The converse holds too:

Proposition 3. $\underline{E}(f) = \lim_n \underline{E}_n(f)$ *for any gamble f.*

Next, we use this Proposition to establish an expression for the natural extension of a generalised p-box in terms of discrete p-boxes. For any natural number $n \geq 1$, and $i \in \{2,\ldots,n\}$, define the sets $A_1^n := \overline{F}^{-1}\left([0,\frac{1}{n}]\right), A_i^n := \overline{F}^{-1}\left((\frac{i-1}{n},\frac{i}{n}]\right), B_1^n := \underline{F}^{-1}\left([0,\frac{1}{n}]\right)$ and $B_i^n := \underline{F}^{-1}\left((\frac{i-1}{n},\frac{i}{n}]\right)$. Clearly, both $\{A_1^n,\ldots,A_n^n\}$ and $\{B_1^n,\ldots,B_n^n\}$ are partitions of Ω. Define $\underline{F}_n$ and $\overline{F}_n$ by

$$\overline{F}_n(x) = \frac{i}{n} \text{ if } x \in A_i^n, \quad \underline{F}_n(x) = \begin{cases} \frac{i-1}{n} & \text{if } x \in B_i^n \text{ and } x \neq 1_\Omega, \\ 1 & \text{if } x = 1_\Omega. \end{cases} \tag{5}$$

Lemma 3. *The following statements hold for all $x \in \Omega$:*

(i) For any $n \in \mathbb{N}$, $\underline{F}_n$ and $\overline{F}_n$ are cdfs, $\underline{F}_n(x) \leq \underline{F}(x)$, and $\overline{F}(x) \leq \overline{F}_n(x)$.
(ii) $\lim_n \underline{F}_n(x) = \underline{F}(x)$ and $\lim_n \overline{F}_n(x) = \overline{F}(x)$.
(iii) $(\underline{F}_{2^n})_n$, $(\overline{F}_{2^n})_n$ are increasing and decreasing sequences of cdfs such that $\underline{F}(x) = \lim_n \underline{F}_{2^n}(x)$ and $\overline{F}(x) = \lim_n \overline{F}_{2^n}(x)$.

If we can find a simple expression for the natural extension of $\underline{P}_n$ for our particular choice of $\underline{F}_n$ and $\overline{F}_n$, then we also have a simple expression for $\underline{E}_{\underline{F},\overline{F}}$ via Proposition 3. Consider $\underline{G}_1, \ldots, \underline{G}_n$ and $\overline{G}_1, \ldots, \overline{G}_n$ defined by

$$\underline{G}_i(x) = \begin{cases} 1 & \text{if } \underline{F}_n(x) \geq \frac{i}{n} \\ 0 & \text{otherwise} \end{cases} \qquad \overline{G}_i(x) = \begin{cases} 1 & \text{if } \overline{F}_n(x) \geq \frac{i}{n} \\ 0 & \text{otherwise} \end{cases}$$

Proposition 4. *For each $n \in \mathbb{N}$, $\underline{E}_n = \frac{1}{n}\sum_{i=1}^n \underline{E}_{\underline{G}_i,\overline{G}_i}$.*

Hence, all we need to characterise the natural extension of $(\underline{F}_n, \overline{F}_n)$ is to determine the natural extension of a *degenerate p-box*, i.e. one where the lower and upper cdfs only assume the values 0 and 1. Note that a degenerate p-box $(\underline{G}, \overline{G})$ is uniquely determined by

$$I_{(\underline{G},\overline{G})} = \left\{x \in \Omega \colon \underline{G}(x) < \overline{G}(x)\right\} = \left\{x \in \Omega \colon \underline{G}(x) = 0 \text{ and } \overline{G}(x) = 1\right\}.$$

Proposition 5. *Let $(\underline{G}, \overline{G})$ be degenerate and $f \in \mathscr{L}(\Omega)$. If $0_\Omega \notin I_{(\underline{G},\overline{G})}$,*

(i) If $I_{(\underline{G},\overline{G})} = (a,b)$ then $\underline{E}_{\underline{G},\overline{G}}(f) = \inf_{z \in (a,b]} f(z)$.
(ii) If $I_{(\underline{G},\overline{G})} = (a,b]$ then $\underline{E}_{\underline{G},\overline{G}}(f) = \lim_{y \xrightarrow{>} b} \inf_{z \in (a,y]} f(z)$.
(iii) If $I_{(\underline{G},\overline{G})} = [a,b)$ then $\underline{E}_{\underline{G},\overline{G}}(f) = \lim_{x \xrightarrow{<} a} \inf_{z \in (x,b]} f(z)$.
(iv) If $I_{(\underline{G},\overline{G})} = [a,b]$ then $\underline{E}_{\underline{G},\overline{G}}(f) = \lim_{x \xrightarrow{<} a} \lim_{y \xrightarrow{>} b} \inf_{z \in (x,y]} f(z)$.

On the other hand, if $0_\Omega \in I_{(\underline{G},\overline{G})}$, then

(a) If $I_{(\underline{G},\overline{G})} = [0_\Omega, b)$ then $\underline{E}_{\underline{G},\overline{G}}(f) = \inf_{z \in [0_\Omega,b]} f(z)$.
(b) If $I_{(\underline{G},\overline{G})} = [0_\Omega, b]$ then $\underline{E}_{\underline{G},\overline{G}}(f) = \lim_{y \xrightarrow{>} b} \inf_{z \in [0_\Omega,y]} f(z)$.

Concluding, if we consider now the natural extension $\underline{E}'_n$ of $(\underline{F}_{2^n}, \overline{F}_{2^n})$ as defined in Eq. (5), it follows from Proposition 3 and Lemma 3 that $(\underline{E}'_n)_n$ is an increasing sequence of functionals that converges point-wise to $\underline{E}$. By Proposition 4, $\underline{E}'_n$ can be calculated as a convex combination of natural extensions of degenerate p-boxes, whose expressions follow from Proposition 5.

6 Conclusions

We have extended results concerning p-boxes from finite to infinite sets. In particular, we have proven that the natural extension of a p-box characerizing the coherent extensions to all gambles is a completely monotone lower prevision. Such lower previsions have interesting mathematical properties—i.e., they can be written as a Rieman integral, and are determined by their values on events—and relate to comonotone additive functionals, which are of interest in economics.

A convergence result for generalised p-boxes is given in Section 5: any generalised p-box can be expressed as a limit of a sequence of discrete p-boxes. This is interesting because discrete p-boxes are more manageable in practice, and are also related to earlier works [2, 3]. In particular, they can be related to belief functions and to finitely-valued random sets. Also of interest is that natural extension is preserved when taking point-wise limits of monotone sequences of p-boxes.

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