

Motivations

There exist many practical representations of imprecise probabilities. It is therefore important to study their relationships, for various reasons (compare their expressive power and check how easy they can be handled for propagation, fusion, ...).

Here, we study convex families \mathcal{P} of probability distributions defined over finite spaces $X = \{x_1, \dots, x_n\}$ and s.t. $\mathcal{P} = \{P | \forall A \subseteq X, \underline{P}(A) \leq P(A) \leq \overline{P}(A)\}$ with $\underline{P}(A) = \inf_{P \in \mathcal{P}}(P(A))$, $\overline{P}(A) = \sup_{P \in \mathcal{P}}(P(A))$.

Random Sets [2]

Mapping Γ from probability space to power set $\wp(X)$
For finite X , masses $m \geq 0$ over $\wp(X)$ ($\sum_{E \subseteq X} m(E) = 1$; $m(\emptyset) = 0$)

$$\Rightarrow Bel(A) = \sum_{E, E \subseteq A} m(E); Pl(A) = 1 - Bel(A^c) = \sum_{E, E \cap A \neq \emptyset} m(E).$$

Associated Probability family

$$\Rightarrow \mathcal{P}_{Bel} = \{P | \forall A \subseteq X, Bel(A) \leq P(A) \leq Pl(A)\}$$

$Bel = \underline{P} : \infty$ -monotone capacity

Possibility distributions [3]

Mapping π from X to $[0, 1]$ ($\exists x$ s.t. $\pi(x)$)
 $\Rightarrow \Pi(A) = \sup_{x \in A} \pi(x); N(A) = 1 - \Pi(A^c).$

Associated Probability family

$$\Rightarrow \mathcal{P}_\pi = \{P, \forall A \subseteq X, N(A) \leq P(A) \leq \Pi(A)\}$$

$N = \underline{P} : \infty$ -monotone and maxitive capacity

Probability Intervals [1]

Probability bounds $[l_i, u_i]$ on singletons x_i .

Associated Probability family

$$\mathcal{P}_L = \{P | l_i \leq p(x_i) \leq u_i \forall x_i \in X\}$$

$$\Rightarrow \underline{P}(A) = \max(\sum_{x_i \in A} l_i, 1 - \sum_{x_i \notin A} u_i);$$

$$\overline{P}(A) = \min(\sum_{x_i \in A} u_i, 1 - \sum_{x_i \notin A} l_i)$$

$\underline{P} : 2$ -monotone capacity

P-boxes [4]

Pair of cumulative distributions $F \leq \overline{F}$.

Associated probability family

$$\Rightarrow \mathcal{P}_{F, \overline{F}} = \{P | F(x) \leq F^P(x) \leq \overline{F}(x) \quad \forall x \in \mathbb{R}\} \text{ with } p \text{ prob. dist.}$$

► Notion of cumulative distribution based on natural ordering of numbers

Generalized cumulative distribution

- \leq_R : complete preordering on X and R -downset $(x)_R : \{x_i : x_i \leq_R x\}$
- Generalized cumulative distribution $F_R(x) = \Pr((x)_R) \forall x \in X$

Neumaier's Clouds [5]

Pair of distributions (δ, π) with $\delta(x) \leq \pi(x) \forall x \in X$
equivalent to Interval-valued Fuzzy sets

Associated probability family

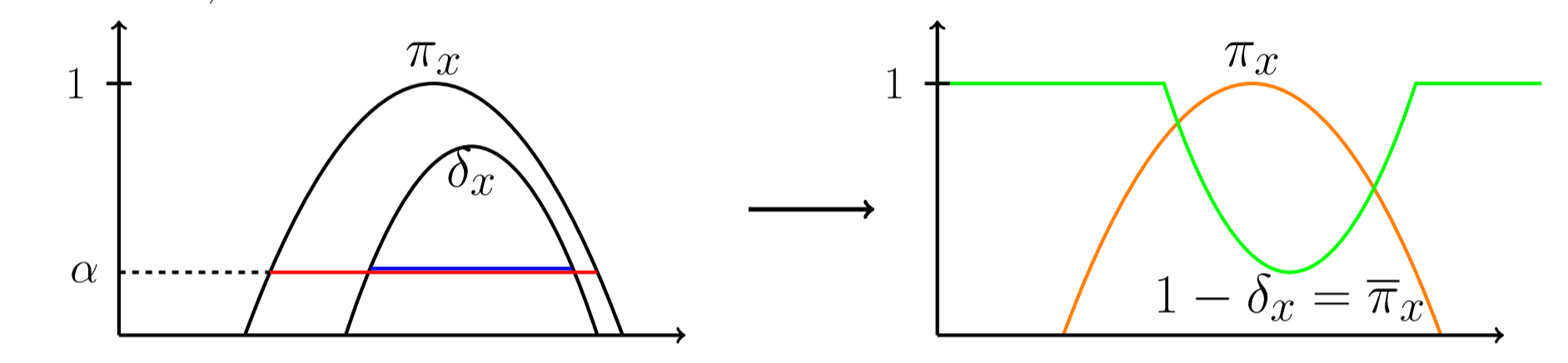
$$\Rightarrow \mathcal{P}_{\delta, \pi} = \{P | P(\{x | \delta(x) \geq \alpha\}) \leq 1 - \alpha \leq P(\{x | \pi(x) > \alpha\})\}$$

Thin clouds : $\delta = \pi$

Fuzzy clouds : $\delta(x) = 0 \forall x \in X$
(equivalent to possibility distributions)

Relation with possibility distributions

$$\Rightarrow \mathcal{P}_{\delta, \pi} = \mathcal{P}_\pi \cap \mathcal{P}_{1-\delta} \text{ with } 1-\delta, \pi \text{ possibility dist.}$$



Generalized P-boxes

pair of generalized cumulative distributions $F_R(x) \leq \overline{F}_R(x)$

Associated probability family

$$\Rightarrow \mathcal{P}_{F_R, \overline{F}_R} = \{P | F_R(x) \leq F^P(x) \leq \overline{F}_R(x) \quad \forall x \in X\} \text{ with } p \text{ prob. dist.}$$

Relation with possibility distributions

- $F_R(x)$ can be viewed as possibility dist. $\pi_R (\max_{x \in A} F_R^\lambda(x) \geq \Pr(A))$
- $\Rightarrow \mathcal{P}_{F_R, \overline{F}_R} = \mathcal{P}_{F_R} \cap \mathcal{P}_{1-\overline{F}_R}$ with $F_R, 1-\overline{F}_R$ possibility distributions

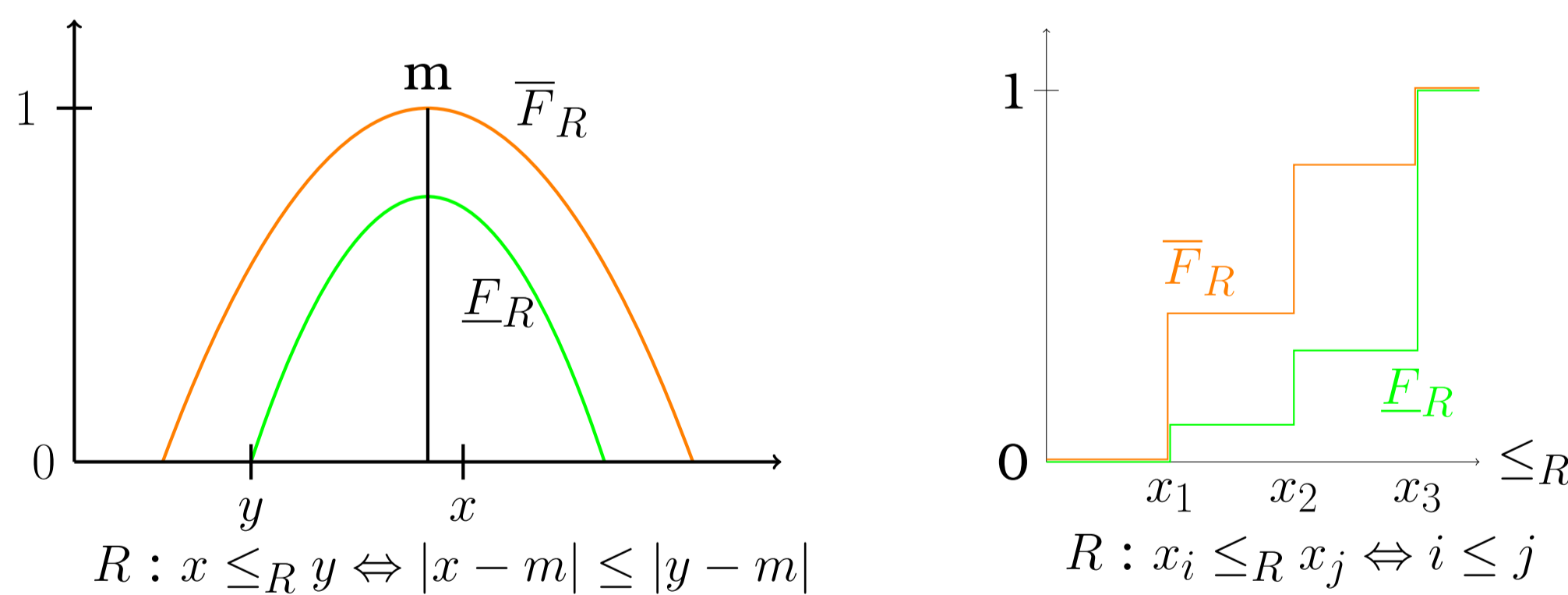
Relation with random sets

$$\Rightarrow \mathcal{P}_{F_R, \overline{F}_R} : \infty\text{-monotone capacity (Gen. p-boxes } \subset \text{ random sets)}$$

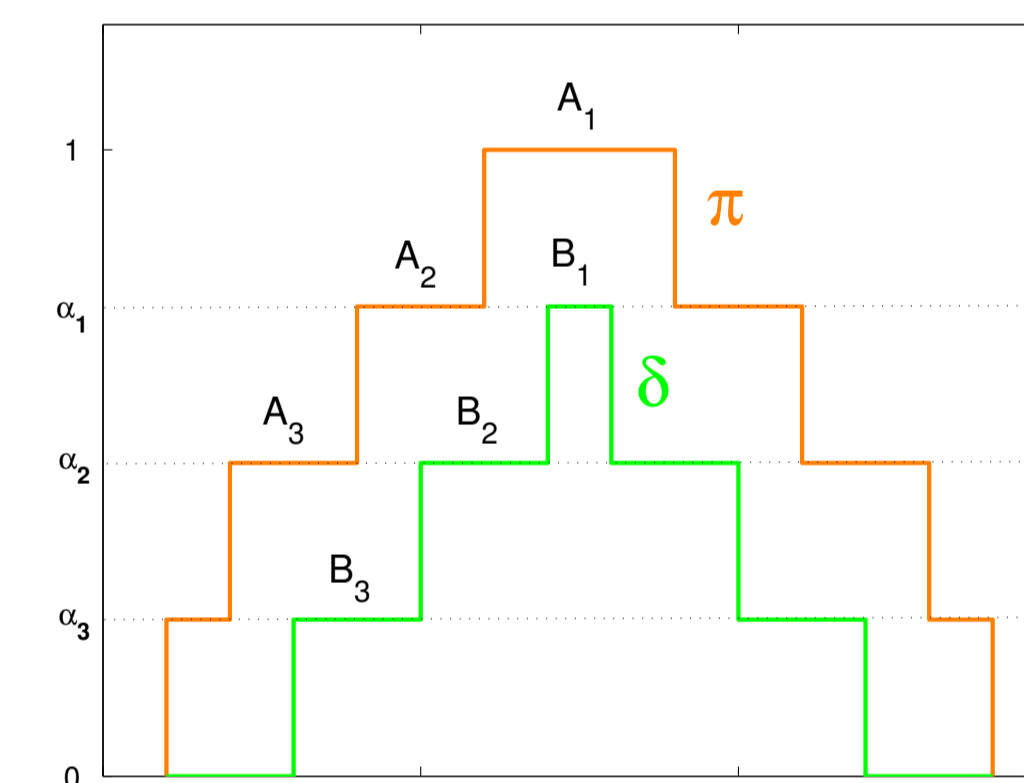
Relation with probability intervals

From $[l_i, u_i]$, compute $F_R(x) = \max(\sum_{x_i \in (x)_R} l_i, 1 - \sum_{x_i \notin (x)_R} u_i)$ and
 $\overline{F}_R(x) = \min(\sum_{x_i \in (x)_R} u_i, 1 - \sum_{x_i \notin (x)_R} l_i) \Rightarrow$ Loss of information

Examples of generalized p-boxes



Comonotonic clouds



Distributions δ, π are comonotonic \Rightarrow sets A_i, B_i are nested

Relation with Gen. p-boxes

Gen. p-boxes and comonotonic clouds are equivalent representations

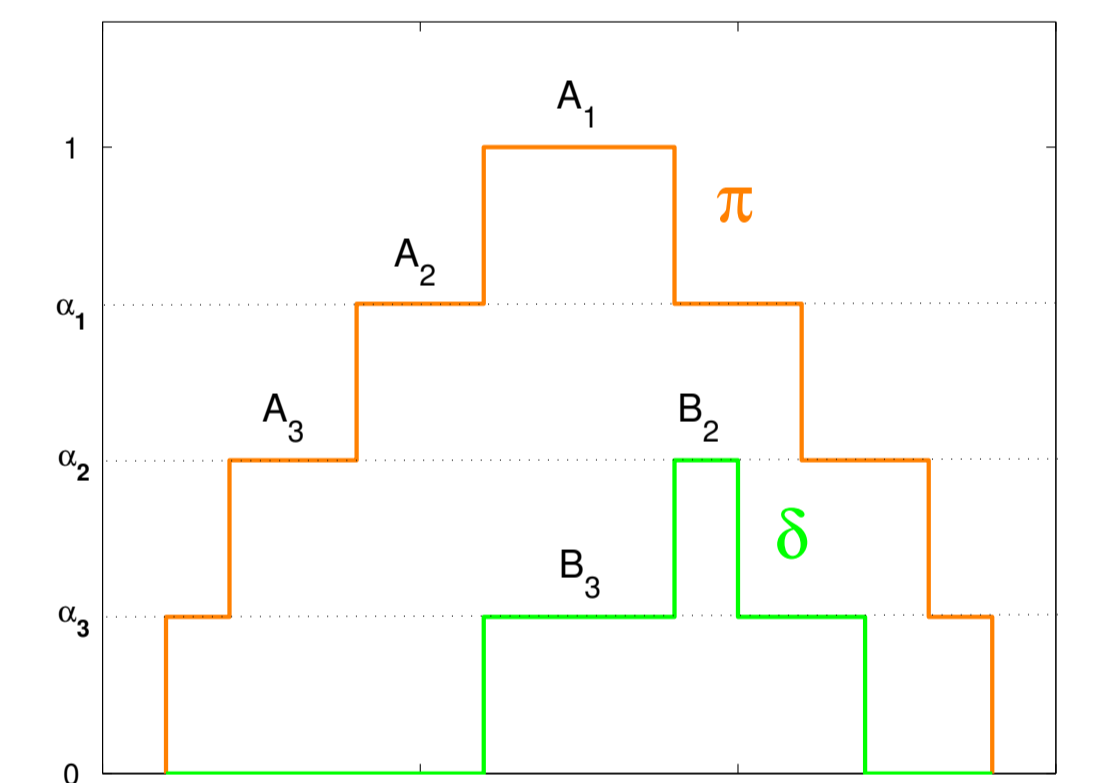
Relation with random sets

$$\Rightarrow \mathcal{P}_{\delta, \pi} : \infty\text{-monotone capacity}$$

Particular cases

Thin clouds : $\mathcal{P}_{\delta, \pi}$ is empty in finite case and contains an ∞ number of distributions in the continuous case.

Non-comonotonic clouds



Distributions δ, π are **not** comonotonic \Rightarrow sets A_i, B_i are no longer nested.

Characterization

If there are 2 sets A_i, B_j s.t. $A_i \cap B_j \notin \{A_i, B_j, \emptyset\}$, then $\Rightarrow \mathcal{P}_{\delta, \pi}$ is not even a 2-monotone capacity.

Neumaier's outer approximation

$$\max(N_\pi(A), N_{1-\delta}(A)) \leq P(A) \leq \min(\Pi_\pi(A), \Pi_{1-\delta}(A))$$

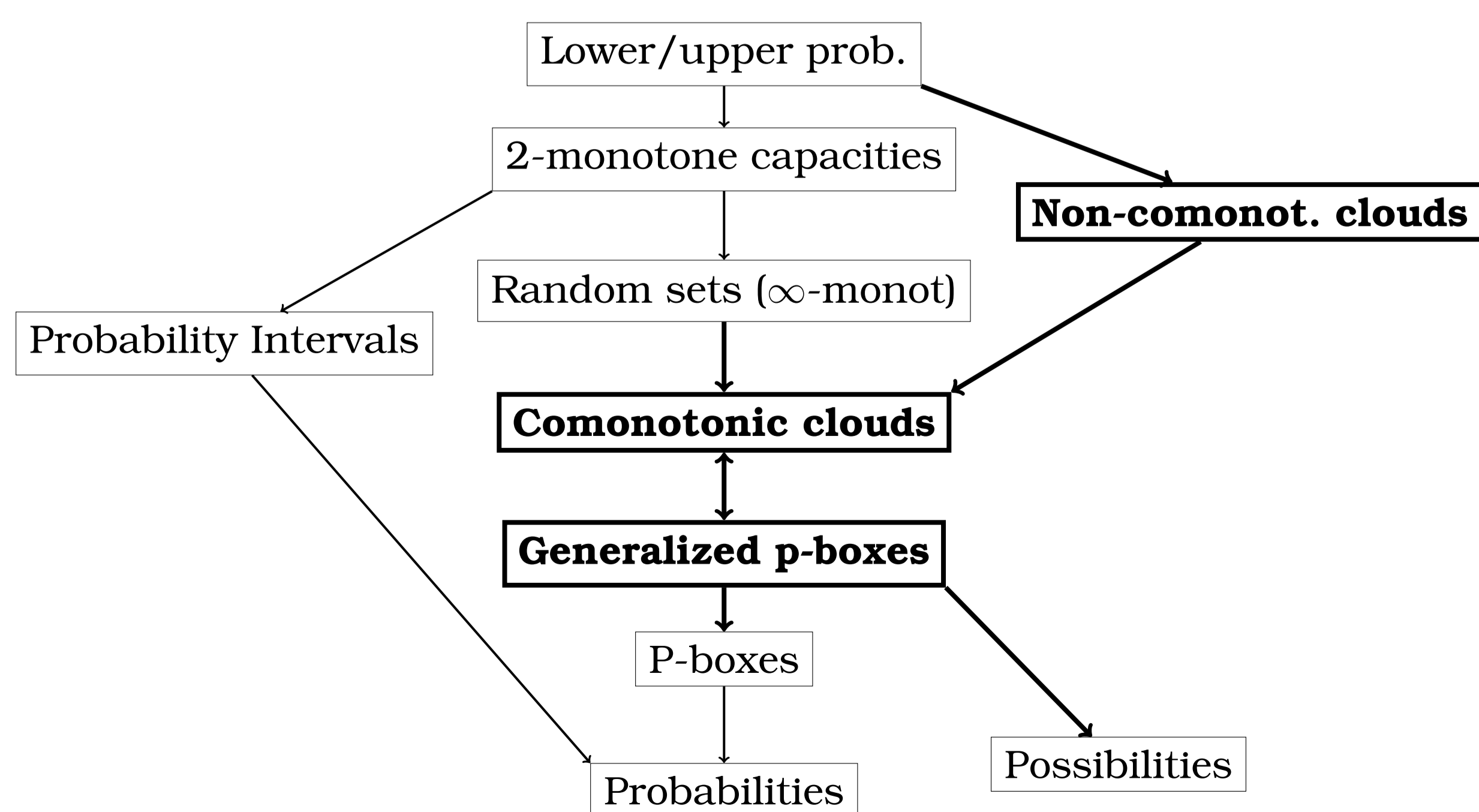
► Bounds easy to compute

Random set inner approximation

Take $m(A_i \setminus B_{i-1}) = \alpha_{i-1} - \alpha_i$

► Advantages of ∞ -monotonicity, exact representation when δ, π are comonotonic.

Summary



Perspectives and open problems

- Extend results to characterize lower/upper previsions of p-boxes, clouds and generalized p-boxes and give formulas of these previsions. (Miranda et al., 2006; de Cooman et al. 2006)
- Are operations of fusion, propagation, computations of expectation and variance, conditioning easy to achieve for these representations?
- Explore link between generalized p-boxes, clouds and linguistic assessments? (de Cooman, 2005)

References

- [1] L. de Campos, J. Huete, and S. Moral. Probability intervals : a tool for uncertain reasoning. *I. J. of Uncertainty, Fuzziness and Knowledge-Based Systems*, 2 :167–196, 1994.
- [2] A. Dempster. Upper and lower probabilities induced by a multivalued mapping, *Annals of Mathematical Statistics* 38 (1967) 325–339.
- [3] D. Dubois, H. Prade, Possibility Theory : An Approach to Computerized Processing of Uncertainty, Plenum Press, 1988.
- [4] S. Ferson, L. Ginzburg, V. Kreinovich, D. Myers, K. Sentz, Constructing probability boxes and dempster-shafer structures, Tech. rep., Sandia National Laboratories (2003).
- [5] A. Neumaier. Clouds, fuzzy sets and probability intervals. *Reliable Computing*, 10 :249–272, 2004.