Cautious conjunctive merging of belief functions

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Problem statement

Merging multiple belief functions

- Information from multiple sources modeled by belief functions
- If possible, merge conjunctively into a single belief function:
  - If sources can be judged independent $\Rightarrow$ use "Dempster's rule"
  - If independence assumption unrealistic $\Rightarrow$ cautious merging rule is one solution

Principle of cautious conjunctive merging

Keep as much information as possible (conjunctive) from each source while adding as few additional assumptions as possible (cautious).

Belief functions formalism

Basic belief assignment (bba)

- X finite space with elements $x_1, \ldots, x_{|X|}$
- bba: function $m : 2^{|X|} \to [0,1]$ s.t. $m(\emptyset) = 0$ and $\sum_{A \subseteq X} m(A) = 1$
- a set A with positive mass $m(A) > 0$ is a focal element

Three measures: Belief, Plausibility, Commonality

- Belief: $bel(E) = \sum_{A \subseteq E} m(A)$
- Plausibility: $pl(E) = \sum_{A \cap E \neq \emptyset} m(A) = 1 - bel(A^c)$
- Commonality: $q(E) = \sum_{E \subseteq A} m(A)$

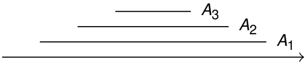
Belief function as a probability family

bba m induces $\mathcal{P}_m = \{P | \forall A \subset X,\ Bel(A) \leq P(A) \leq Pl(A)\}$

Two special kinds of bbas

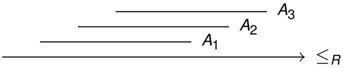
Possibility distributions

- Mapping $\pi : X \to [0,1]$ and $\exists x \in X$ s.t. $\pi(x) = 1$
- Possibility measure: $\Pi(A) = \sup_{x \in A} \pi(x)$
- Necessity measure: $N(A) = 1 - \Pi(A^c)$
- Equivalent to random set with nested focal elements
- $\Pi(A) = Pl(A)$ and $N(A) = Bel(A)$



Generalized p-boxes

- Two comonotone funct. $\underline{F}, \overline{F}$ on X inducing a weak order R: $\overline{F}(x_i) \leq \overline{F}(x_j) \to x_i \leq_R x_j$
- $\exists$ $\overline{x}$ s.t. $\overline{F}(\overline{x}) = 1$, $\underline{x}$ s.t. $\underline{F}(\underline{x}) = 0$
- $\underline{F}(x) = Bel(\{x_i \leq_R x\})$, $\overline{F}(x) = Pl(\{x_i \leq_R x\})$
- $A_i = \{x^i_{inf}, \ldots, x^i_{sup}\}_{\leq_R}$ and $A_j = \{x^j_{inf}, \ldots, x^j_{sup}\}_{\leq_R}$ two distinct focal sets of a bba m. Then, m is a gen p-box iff $(x^i_{inf} \leq_R x^j_{inf}$ and $x^i_{sup} \leq_R x^j_{sup})$ or $(x^i_{inf} \geq_R x^j_{inf}$ and $x^i_{sup} \geq_R x^j_{sup})$ $\forall A_i, A_j \Rightarrow$ focal sets are "shifted" with respect to R



Compare informative contents of bbas

Three usual information orderings of bbas

$m_1 \sqsubseteq_x m_2$: m_1 more x-committed than m_2

- pl-ordering: if $pl_1(A) \leq pl_2(A)\ \forall A \subseteq X$, we note $m_1 \sqsubseteq_{pl} m_2$
  $m_1 \sqsubseteq_{pl} m_2 \Leftrightarrow \mathcal{P}_{m_1} \subseteq \mathcal{P}_{m_2}$
- q-ordering: if $q_1(A) \leq q_2(A)\ \forall A \subseteq X$, we note $m_1 \sqsubseteq_q m_2$
- s-ordering: if m_1 is a specialization of m_2, we note $m_1 \sqsubseteq_s m_2$

  If m_1, m_2 are weight vectors, then bba m_1 is a specialization of bba m_2 if $\exists$ a stochastic matrix S s.t.

  - $m_1 = S \cdot m_2$
  - $S_{ij} > 0 \Rightarrow A_i \subseteq B_j$
  - $m_2(A)$ "flow downs" to subsets of A in m_1

$m_1 \sqsubseteq_s m_2$ imply both $m_1 \sqsubseteq_{pl} m_2$,$m_1 \sqsubseteq_q m_2$ (but **not** the reverse)

Principles

Given m_1, m_2 and their sets of focal elements $\mathcal{F}_1, \mathcal{F}_2$, the result of conjunctively merging m_1, m_2 is a bba m obtained in 2 steps:

1. Define a joint bba m_{12} s.t. $m_1(A) = \sum_{B \in \mathcal{F}_2} m_{12}(A, B)\ \forall A$ and likewise for m_2 (**Marginal preservation**)
2. $m_{12}(A, B)$ is allocated to, and only to $A \cap B$ (**Conjunctive allocation**)

$\mathcal{M}_X^{m_1 \cap m_2}$: set of conjunctively merged bbas m. Every such bba is a specialization of m_1 and m_2.

3 situations for $\mathcal{M}_X^{m_1 \cap m_2}$

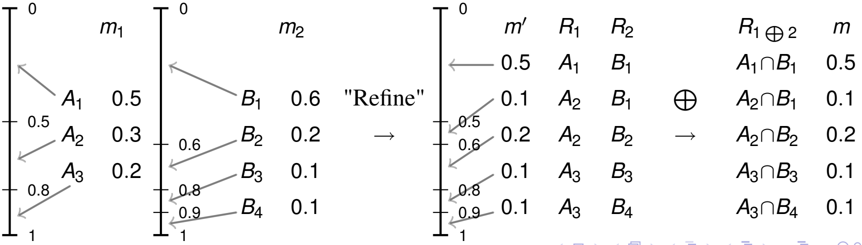
- Either $\forall A \in \mathcal{F}_1, B \in \mathcal{F}_2, A \cap B \neq \emptyset$. m_1, m_2 are said to be **logically consistent** $\Rightarrow \mathcal{M}_X^{m_1 \cap m_2}$ Contains only normalized bbas ($m(\emptyset) = 0$)
- either $\exists A, B$ $A \cap B = \emptyset$ and $\exists$ merged bba m s.t. $m(\emptyset) = 0$ ($\mathcal{P}_{m_1} \cap \mathcal{P}_{m_2} \neq \emptyset$). m_1, m_2 are said to be **non-conflicting** $\Rightarrow$ $\mathcal{M}_X^{m_1 \cap m_2}$ contains both normalized and subnormalized bbas.
- or there is no merged bba m s.t. $m(\emptyset) = 0$ ($\mathcal{P}_{m_1} \cap \mathcal{P}_{m_2} = \emptyset$). m_1, m_2 are said to be **conflicting** $\Rightarrow \mathcal{M}_X^{m_1 \cap m_2}$ contains only subnormalized bbas

Merging with commensurate bbas

Principles

- order focal elements $\mathcal{F}_1, \mathcal{F}_2$ of m_1, m_2
- bbas $(\mathcal{F}_1, m_1)$ and $(\mathcal{F}_2, m_2)$ form two partitions of the unit interval
- take the coarsest common partition refining these two ones, then take conjunctive allocation for each element of this partition.
- result $\in \mathcal{M}_X^{m_1 \cap m_2}$ depend of chosen ordering of focal elements

Illustration



Merging with equi-commensurate bbas

Principle

Take a refinement such that all weights are equal

Illustration

m'	R_1	R_2		
0.5	A_1	B_1		5 lines with m=0.1
0.1	A_2	B_1	"Equi-comm."	
0.2	A_2	B_2	$\longrightarrow$	2 lines with m=0.1
0.1	A_3	B_3		
0.1	A_3	B_4		

Result

With weights small enough and proper re-ordering of elements, we can get as close as we want to any bba $\in \mathcal{M}_X^{m_1 \cap m_2}$

Basic principles

Problem

Find a merging rule ($\bigwedge$) resulting in a bba $m \in \mathcal{M}_X^{m_1 \cap m_2}$ that is "least"-committed, here in the sense of maximized expected cardinality.

Basic requirements

- $\bigwedge$ should be idempotent: $\bigwedge(m, m) = m$
- If m_2 is a specialization of m_1, then $\bigwedge(m_1, m_2) = m_2$

$\Rightarrow$ Concern special cases and do not provide general guidelines

Idea

Find the proper ordering of (equi-)commensurate bbas that maximizes expected cardinality.

Main result

A merged bba m having maximal cardinality ($m \in \mathcal{M}_X^{m_1 \cap m_2}$ with $I(m)$ max.) can be built by commensurate merging in which the ordering of focal elements is an extension of partial ordering induced by inclusion (i.e. $A_i \subset A_j \rightarrow A_i < A_j$).

But . . .

. . . Ranking focal el. with respect to inclusion is neither sufficient nor necessary to find m with maximal cardinality

Interest

Practical

Give some first "general" guidelines to combine marginal belief functions to get a merged bba having a maximized expected cardinality.

Theoretical

If marginal belief functions are possibility distributions, using the (complete) order induced by inclusion comes down to apply the well-known minimum rule ($m = \pi_{\text{min}} = \min(\pi_1, \pi_2)$) $\Rightarrow$ coherence of the rule with possibility theory.

Refining by pl- or q-ordering

Multiple merged bba m having maximal cardinality
$\Rightarrow$ discriminate/refining by using pl- or q- ordering.

$\pi_1 = m_1$	
$\{x_1, x_2, x_3\}$	0.5
$\{x_0, x_1, x_2, x_3, x_4\}$	0.5

$\pi_2 = m_2$	
$\{x_3, x_4, x_5\}$	0.5
$\{x_2, x_3, x_4, x_5, x_6\}$	0.5

$$\longrightarrow \mathcal{M}_X^{m_1 \cap m_2} \longrightarrow$$

$C_1 = \pi_{\min}(I(C_1) = 2)$	
$\{x_3\}$	0.5
$\{x_2, x_3, x_4\}$	0.5

$C_2(I(C_2) = 2)$	
$\{x_3, x_4\}$	0.5
$\{x_2, x_3\}$	0.5

- $C_1 \sqsubset_{pl} C_2$: C_2 least pl-committed (more coherent with probabilistic interpretation, since $\mathcal{P}_{C_1} \subset \mathcal{P}_{C_2}$), but commensurate merging giving C_2 do not respect inclusion order.
- $C_2 \sqsubset_q C_1$: C_1 least q-committed (more coherent with TBM interpretation, possibility theory and proposed rule)

Minimizing conflict

If m_1, m_2 are not logically consistent, maximizing expected cardinality do not in general minimize conflict ($m \in \mathcal{M}_X^{m_1 \cap m_2}$ s.t. $m(\emptyset)$ is minimal). To min. conflict, Cattaneo (2003) proposes to find m that maximizes:

$$F(m) = m(\emptyset)f(0) + (1 - m(\emptyset)) \sum_{A \neq \emptyset} m(A) log_2(A)$$

where $f(0)$ penalizes appearance of conflict. Similar idea can be used with expected cardinality, but then previous results no longer hold.

$\pi_1 = m_1$	
$\{x_1, x_2\}$	0.5
$\{x_0, x_1, x_2, x_3, x_4\}$	0.5

$\pi_2 = m_2$	
$\{x_4\}$	0.5
$\{x_2, x_3, x_4, x_5, x_6\}$	0.5

$\xrightarrow{m(\emptyset) \text{ min.}}$

C_1	
$\{x_2\}$	0.5
$\{x_4\}$	0.5

Least-commitment and weight functions

(Denoeux, 2007) proposes a cautious rule based on an ordering (w-ord.) induced by canonical decompostion of bba (Smets, 1995).

advantages

- Uniqueness of the solution
- Operationally very convenient
- Associative and commutative

drawbacks

- Restriction of possible joint bbas to a subset of $\mathcal{M}_X^{m_1 \cap m_2}$
- Not coherent with minimum of possibility theory
- Difficult to compare with notions using s-ordering

Conclusions/Perspectives

Conclusions

We studied cautious merging consisting in maximizing expected cardinality:

- First general and practical guidelines using commensurate bbas and inclusion ordering between focal el. to perform the merging
- Coherent with notion of cautiousness in possibility theory
- Compete with other propositions

Perspectives

- Add constraints/guidelines to have sufficient conditions to reach maximized exp. card. (increase efficiency)
- Pursue the comparison between maximization of exp. card. and other notions of least-commitment
- Check for associativity/commutativity in the general case