# Relating practical representations of imprecise probabilities

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# Introducing Didier

## Position

CNRS Research advisor, IRIT, Toulouse

## Some interests and hobbies

- Uncertainty treatment, applied mathematics, artificial intelligence, qualitative and quantitative possibility theory, . . .
- Singing
- Proposing ideas to his Phd students
- . . .

## Collaborations

- Quite a few!

# Introducing Eric

## Position

Research engineer, IRSN, Cadarache, France

## (Research) interests

Applying imprecise probabilities, Dempster-Shafer theory, fuzzy calculus to

- Radiological protection
- Environmental issues
- Nuclear safety

# Introducing me (again)

## Position

Phd student at the Institute of radiological protection and nuclear safety, under the supervision of Didier Dubois (IRIT) and Eric Chojnacki (IRSN)

## Main interests

Treatment of information in uncertainty analysis, using imprecise models

- Information modeling
- Information fusion
- (In)dependence concepts
- Propagation of information

# Introducing the center

21 km$^2$, 40 km from any middle-sized city.

# Introducing the boars

# Outline

Family $\mathcal{P}$ of probabilities can be hard to represent (even by lower ($\underline{P}(A)$) and upper ($\overline{P}(A)$) probabilities). Special cases easier to handle exist :

- Random sets
- Possibility distributions
- (Generalized) P-boxes
- Neumaier's Clouds
- Probability intervals

# Possibility formalism

## Definition

- Mapping $\pi : X \rightarrow [0,1]$ and $\exists x \in X$ s.t. $\pi(x) = 1$
- Possibility measure: $\Pi(A) = \sup_{x \in A} \pi(x)$ (maxitive)
- Necessity measure: $N(A) = 1 - \Pi(A^c)$

## Possibility and random sets

Possibility distribution $=$ random set with nested realizations

## Probability family associated to possibility distribution

$\mathcal{P}_\pi = \{P | \forall A \subseteq X \text{ measurable}, N(A) \leq P(A) \leq \Pi(A)\}$

# Generalized cumulative distribution

## (Generalized) Cumulative Distribution

A Cumulative distribution is a monotone function $F$ from a weakly ordered space $X$ to $[0,1]$, with $F(\overline{x}) = 1$ ($\overline{x} =$ top of $X$).

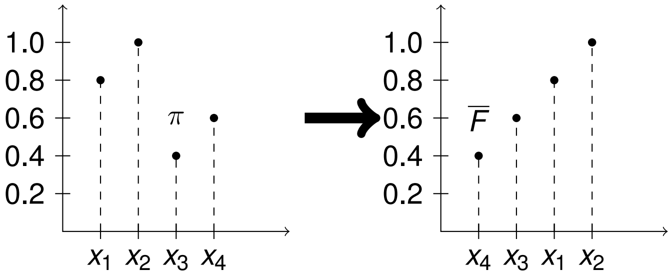
- Usual dist : $F(x)=Pr((-\infty,x])$ — $X =$ reals; Order = Natural ordering of numbers
- Gen. dist : $F(x)=Pr(\{x_i \in X | x_i \leq_R x\})$ — $X =$ arbitrary space; Order R = any weak order over $X$

## Link with possibility distributions

an **upper** cumulative distribution $\overline{F}$ bounding a probability family is such that $\max_{x \in A} \overline{F}(x) \geq \Pr(A)$ (maxitivity), and can thus be interpreted as a possibility distribution $\pi$

# Generalized cumulative distribution

Up to a re-ordering, any possibility distribution $\pi$ can be assimilated to an upper (generalized) cumulative distribution $\overline{F}$.



# Generalized P-boxes

## (Generalized) P-box

A (generalized) P-box is a pair of comonotone functions $\underline{F}, \overline{F}$ from $X$ to $[0,1]$, with $\underline{F}(x) \leq \overline{F}(x)$ and $\exists \ \overline{x}$ s.t. $\overline{F}(\overline{x}) = 1, \exists \ \underline{x}$ s.t. $\underline{F}(\underline{x}) = 0$

**Associated probability family**

► $\mathcal{P}_{p-box} = \{P | \underline{F}(x) \leq P(\{x_i \in X | x_i \leq_R x\}) \leq \overline{F}(x)\}$ with $R$ a weak order on $X$

# Generalized P-boxes : constraint view

(Generalized) p-boxes can be viewed as upper and lower uncertainty bounds on nested confidence sets induced by the weak order $R$ (A similar view for usual p-boxes is adopted by I. Kozine, L. Utkin (I.J. of Gen. Syst., 2005) )

- Let $A_i = \{x \in X | x \leq_R x_i\}$ with $x_i \leq_R x_j$ iff $i < j$
- $A_1 \subset A_2 \subset \ldots \subset A_n$
- Gen. P-box can be encoded by following constraints :

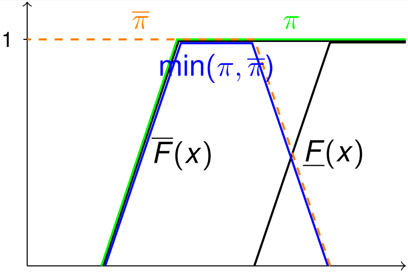
$$\alpha_i \leq P(A_i) \leq \beta_i \qquad i = 1, \ldots, n$$
$$\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n \leq 1$$
$$\beta_1 \leq \beta_2 \leq \ldots \leq \beta_n \leq 1$$

# Illustration

## Link with possibility distributions

If $F_*(x)$ is a lower generalized cumulative distribution, we have $\min_{x \in A^c} F_*(x) \leq \Pr(A) \rightarrow \max_{x \in A^c}(1 - F_*(x)) \geq \Pr(A^c)$.

- Take $\pi = \overline{F}(x)$, $\overline{\pi} = 1 - \underline{F}(x)$, we have $\mathcal{P}_{p-box} = \mathcal{P}_{\pi} \cap \mathcal{P}_{\overline{\pi}}$

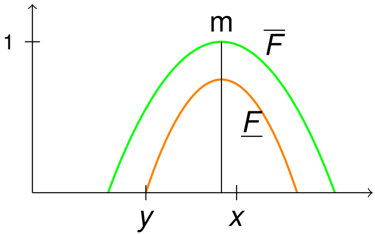



$$(\mathcal{P}_{p-box} = \mathcal{P}_{\pi} \cap \mathcal{P}_{\overline{\pi}}) \supset (\mathcal{P}_{\min(\pi,\overline{\pi})})$$

# Generalized P-box : a surprising example

## A funny example

This is not an usual p-box, but it is a generalized p-box!




- m : mode of the two distributions
- R : $x \leq_R y \Leftrightarrow |x - m| \leq |y - m|$

# Neumaier's clouds : Introduction

## Definition

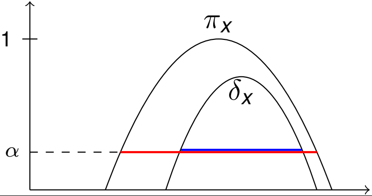
A cloud can be viewed as a pair of distributions $[\delta(x) \leq \pi(x)]$ from $X$ to $[0, 1]$ ($\equiv$ to an interval-valued fuzzy set)

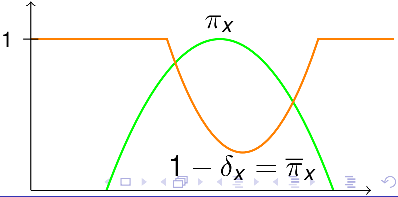
**Associated probability family**

- $\mathcal{P}_{cloud} = \{P | P(\{x|\delta(x) \geq \alpha\}) \leq 1 - \alpha \leq P(\{x|\pi(x) > \alpha\})$

**Link with possibility distributions**

- If we consider the possibility distributions $1 - \delta = \overline{\pi}$ and $\pi$, we have $\mathcal{P}_{cloud} = \mathcal{P}_{\pi} \cap \mathcal{P}_{1-\delta=\overline{\pi}}$ (Dubois & Prade 2005)



# Discrete clouds : formalism

## Discrete clouds as collection of sets

Discrete clouds can be viewed as two collections of confidence sets

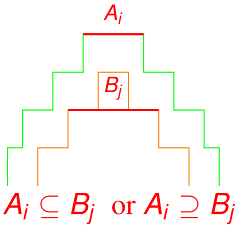
- $\emptyset = A_0 \subset A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \subset A_{n+1} = X \quad (\pi_x)$
- $\emptyset = B_0 \subset B_1 \subseteq B_2 \subseteq \ldots \subseteq B_n \subset B_{n+1} = X \quad (\delta_x)$
- $B_i \subseteq A_i \quad (\delta_x \leq \pi_x)$

with constraints

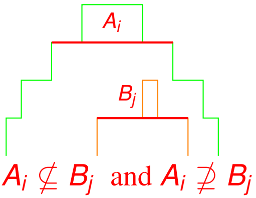
- $P(B_i) \leq 1 - \alpha_i \leq P(A_i)$
- $1 = \alpha_0 > \alpha_1 > \alpha_2 > \ldots > \alpha_n > \alpha_{n+1} = 0$

## Characterizing clouds

- A cloud is said comonotonic if distributions $\delta, \pi$ are comonotone
- A cloud is said non-comonotonic if distributions $\delta, \pi$ are **not** comonotone




Comonotonic cloud




Non-comonotonic cloud

# Main results on clouds and gen. p-boxes

## Comonotonic clouds

- Gen. p-boxes and comonotonic clouds are equivalent representations
- Comonotonic clouds induce $\infty$-monotone capacities, and are thus a special case of random sets.

## Non-comonotonic clouds

- Non-comonotonic clouds are not even 2-monotone capacities
- Neumaier's outer approximation : $\max(N_\pi(A), N_{1-\delta}(A)) \leq P(A) \leq \min(\Pi_\pi(A), \Pi_{1-\delta}(A))$
- Random set inner approximation : $m(A_i \setminus B_{i-1}) = \alpha_{i-1} - \alpha_i$ (exact when $\delta, \pi$ are comonotonic).

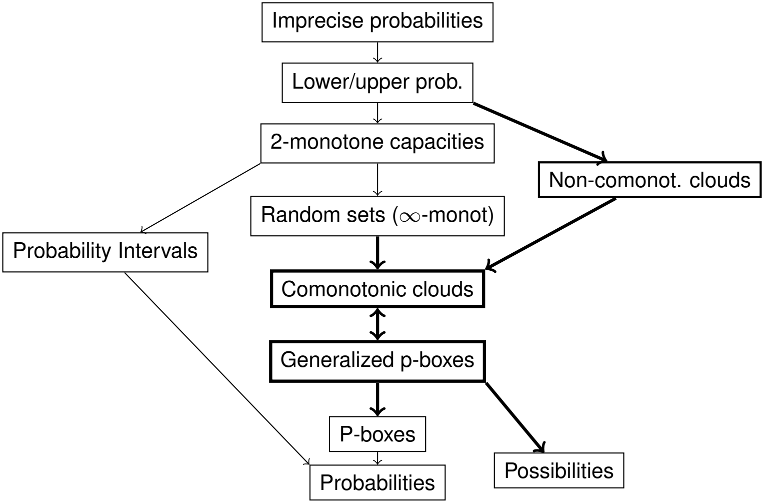
# Relations with probability intervals

Probability intervals are imprecise probability assignments to elements in a finite set. They induce lower probabilities that are 2-monotone capacities.
Since clouds (Comonotonic or not) can be seen as imprecise probability assignments to confidence intervals, transforming one of the two representations into the other implies:

- Either losing information (i.e. by building an outer approximation of the original representation)
- Or adding information (i.e. by building an inner approximation of the original representation)

► Defining a systematic transformation that loses (adds) a minimal amount of information is an open problem.

# Imprecise probability representations: where is what?



# Perspectives and open questions

- Study the propagation, fusion, conditioning of generalized p-boxes and clouds (are they easy to compute, do they preserve the representation ?).
- Extend results to characterize lower/upper previsions of clouds and generalized p-boxes (in progress...)
- Explore the link that could exist between Gen. p-boxes, clouds (i.e. pairs of possibility distributions) and linguistic assessments of imprecise probabilities.