

On the relationships between random sets, possibility distributions, p-boxes and clouds

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Outline

Family \mathcal{P} of probabilities can be hard to represent (even by lower ($\underline{P}(A)$) and upper ($\overline{P}(A)$) probabilities). Special cases easier to handle exist :

- 1 Random Sets and Possibility distributions
- 2 P-Boxes
- 3 Clouds

Outline

- 1 Random Sets and Possibility distributions
- 2 P-Boxes
 - Generalized P-Boxes
 - Relationships between P-Boxes and random sets
 - Relationships between P-Boxes and possibility distribution
- 3 Clouds
 - Clouds formalism
 - Relations and characterization
 - Approximating clouds

Random Sets formalism

Definition

- Multi-valued mapping from probability space to space X
- Here, mass function $m : 2^X \rightarrow [0, 1]$ and $\sum_{E \subseteq X} m(E) = 1$
- A set $E \subseteq X$ is a focal set iff $m(E) > 0$
- Belief function : $Bel(A) = \sum_{E, E \subseteq A} m(E)$
- Plausibility function : $Pl(A) = \sum_{E, E \cap A \neq \emptyset} m(E)$

Probability family induced by random sets

$$\mathcal{P}_{Bel} = \{P \mid \forall A \subseteq X \text{ measurable, } Bel(A) \leq P(A) \leq Pl(A)\}$$

Possibility formalism

Definition

- Mapping $\pi : X \rightarrow [0, 1]$ and $\exists x \in X$ s.t. $\pi(x) = 1$
- Possibility measure: $\Pi(A) = \sup_{x \in A} \pi(x)$
- Necessity measure: $N(A) = 1 - \Pi(A^c)$

Possibility and random sets

Possibility distribution = random set with nested focal elements

Probability family induced by possibility distribution

$$\mathcal{P}_\pi = \{P \mid \forall A \subseteq X \text{ measurable, } N(A) \leq P(A) \leq \Pi(A)\}$$

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Generalized cumulative distribution

Usual cumulative distribution

Let Pr be a probability function on \mathbb{R} : the cumulative distribution is $F(x) = \Pr((-\infty, x])$

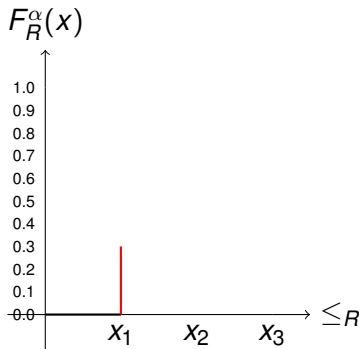
Preliminary definitions

- Let X be a finite domain of n elements and $\alpha = (\alpha_1 \dots \alpha_n)$ a probability distribution
- R is a relation defining a complete pre-ordering \leq_R on X
- a R -downset $(x]_R$ contains every element x_i s.t. $x_i \leq_R x$

Definition

Given a relation R , a generalized cumulative distribution is defined as $F_R^\alpha(x) = \Pr((x]_R)$.

Generalized cumulative distribution : illustration



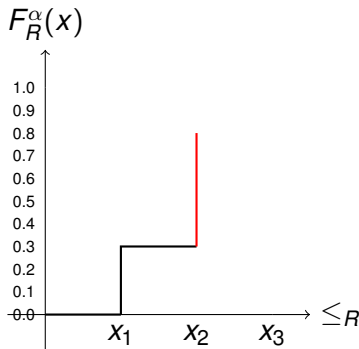
example

- $X = \{x_1, x_2, x_3\}$
- $\alpha = \{0.3, 0.5, 0.2\}$
- $R : x_i < x_j$ iff $i < j$
- $X_R = \{x_1, x_2, x_3\}$

Cumulative prob.

- $F_R^\alpha(x_1) = P(x_1) = 0.3$
- $F_R^\alpha(x_2) = P(x_1, x_2) = 0.8$
- $F_R^\alpha(x_3) = P(x_1, x_2, x_3) = 1$

Generalized cumulative distribution : illustration



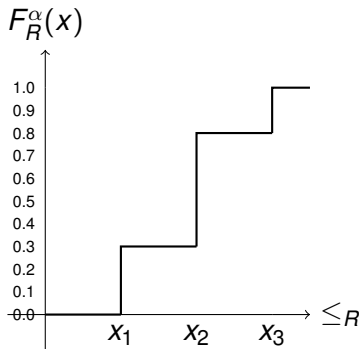
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Generalized cumulative distribution : illustration



example

- $X = \{x_1, x_2, x_3\}$
- $\alpha = \{0.3, 0.5, 0.2\}$
- $R : x_i < x_j \text{ iff } i < j$
- $X_R = \{x_1, x_2, x_3\}$

Cumulative prob.

- $F_R^\alpha(x_1) = P(x_1) = 0.3$
- $F_R^\alpha(x_2) = P(x_1, x_2) = 0.8$
- $F_R^\alpha(x_3) = P(x_1, x_2, x_3) = 1$

Generalized P-boxes : definition

Usual P-boxes

A P-box is a pair of cumulative distributions $(\underline{F}, \overline{F})$ bounding an imprecisely known distribution F ($\underline{F} \leq F \leq \overline{F}$)

Definition

Given R , a generalized p-box is a pair of gen. cumulative distributions $(F_R^\alpha(x) \leq F_R^\beta(x))$ bounding an imprecisely known distribution $F_R(x)$

Probability family induced by generalized p-box

$$\mathcal{P}_{p\text{-box}} = \{P | \forall x \in X \text{ measurable, } F_R^\alpha(x) \leq F_R(x) \leq F_R^\beta(x)\}$$

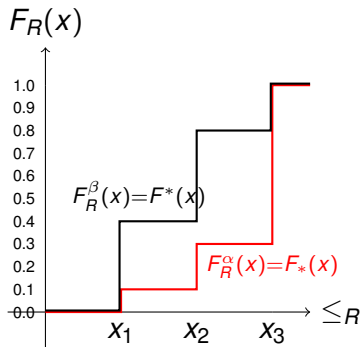
Generalized P-boxes : constraint representation

To define a complete pre-order R between elements x_i is equivalent to define a sequence of nested confidence sets.

- Let $A_i = (x_i]_R$ with $x_i \leq_R x_j$ iff $i < j$
- $A_1 \subset A_2 \subset \dots \subset A_n$
- Gen. P-box can be encoded by following constraints :

$$\begin{aligned}\alpha_i &\leq P(A_i) \leq \beta_i & i = 1, \dots, n \\ \alpha_1 &\leq \alpha_2 \leq \dots \leq \alpha_n \leq 1 \\ \beta_1 &\leq \beta_2 \leq \dots \leq \beta_n \leq 1\end{aligned}$$

Generalized P-box : illustration



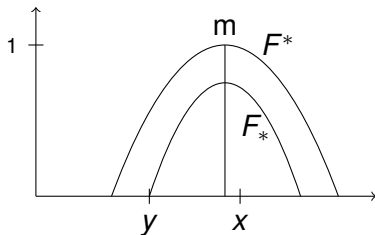
constraints

- $0.1 \leq P(A_1) = P(x_1) \leq 0.4$
- $0.3 \leq P(A_2) = P(x_1, x_2) \leq 0.8$
- $1 \leq P(A_3) = P(x_1, x_2, x_3) \leq 1$

Generalized P-box : another illustration

A more surprising gen. p-box

This also defines a generalized p-box.



- m : mode of the two distributions
- $R : x \leq_R y \Leftrightarrow |x - m| \leq |y - m|$

Random sets/P-boxes relation

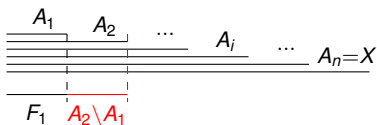
Theorem

Any generalized p-box is a special case of random set (there is a random set such that $\mathcal{P}_{Bel} = \mathcal{P}_{p\text{-box}}$)

Sketch of proof

Lower probabilities on every possible event are the same in the two cases

P-Box \rightarrow random set algorithm

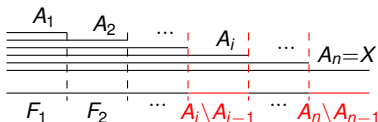


algorithm

- 1 Build partition of X
- 2 Order α_i, β_i and rename them γ_I
- 3 Build focal sets E_i with weights

$$m(E_I) = \gamma_I - \gamma_{I-1}$$

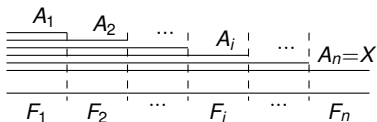
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P-Box \rightarrow random set algorithm



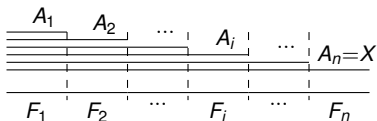
$$\alpha_0 = \beta_0 = 0 \leq \alpha_1 \leq \dots \leq \beta_n \leq 1 = \beta_{n+1} = \alpha_{n+1}$$

$$\alpha_0 = \gamma_0 = 0 \leq \gamma_1 \leq \dots \leq \gamma_{2n} \leq 1 = \gamma_{2n+1} = \beta_{n+1}$$

algorithm

- 1 Build partition of X
- 2 Order α_i, β_i and rename them γ_l
- 3 Build focal sets E_i with weights
 $m(E_l) = \gamma_l - \gamma_{l-1}$

P-Box \rightarrow random set algorithm



$$\alpha_0 = \beta_0 = 0 \leq \alpha_1 \leq \dots \leq \beta_n \leq 1 = \beta_{n+1} = \alpha_{n+1}$$

$$\alpha_0 = \gamma_0 = 0 \leq \gamma_1 \leq \dots \leq \gamma_{2n} \leq 1 = \gamma_{2n+1} = \beta_{n+1}$$

$$m(E_l) = \gamma_l - \gamma_{l-1}$$

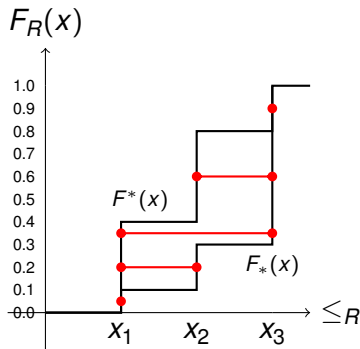
$$\text{with } E_l = E_{l-1} \cup F_{i+1} \text{ if } \gamma_{l-1} = \alpha_i$$

$$\text{with } E_l = E_{l-1} \setminus F_i \text{ if } \gamma_{l-1} = \beta_i$$

algorithm

- 1 Build partition of X
- 2 Order α_i, β_i and rename them γ_l
- 3 Build focal sets E_i with weights $m(E_i) = \gamma_l - \gamma_{l-1}$

graphical representation



Random Set

- $m(E_1) = m(\{x_1\}) = 0.1$
- $m(E_2) = m(\{x_1, x_2\}) = 0.2$
- $m(E_3) = m(\{x_1, x_2, x_3\}) = 0.1$
- $m(E_4) = m(\{x_2, x_3\}) = 0.4$
- $m(E_5) = m(\{x_3\}) = 0.2$

Generalized cumulative distribution

An upper generalized cumulative distribution $F^*(x)$ can be viewed as a possibility distribution π , since

$\max_{x \in A} F^*(x) \geq \Pr(A)$. If $F_*(x)$ is a lower generalized cumulative distribution, we have

$$\min_{x \in A^c} F_*(x) \leq \Pr(A) \rightarrow \max_{x \in A^c} (1 - F_*(x)) \geq \Pr(A^c)$$

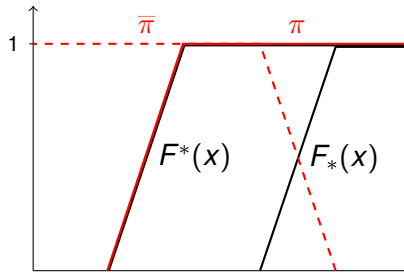
Generalized P-box

- Two generalized cumulative distributions $F^*(x) \geq F_*(x)$
- Let π be a possibility distribution s.t. $\pi = F^*(x)$
- Let $\bar{\pi}$ be a possibility distribution s.t. $\bar{\pi} = 1 - F_*(x)$

Probability families equivalence

We have that $\mathcal{P}_{p\text{-box}} = \mathcal{P}_{\pi} \cap \mathcal{P}_{\bar{\pi}}$

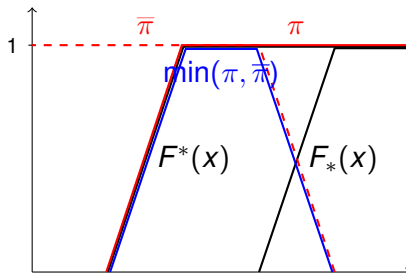
Illustration



Relations

$$(\mathcal{P}_{p\text{-box}} = \mathcal{P}_{\pi} \cap \mathcal{P}_{\bar{\pi}})$$

Illustration



Relations

$$(\mathcal{P}_{p\text{-box}} = \mathcal{P}_{\pi} \cap \mathcal{P}_{\bar{\pi}}) \supset (\mathcal{P}_{\min(\pi, \bar{\pi})})$$

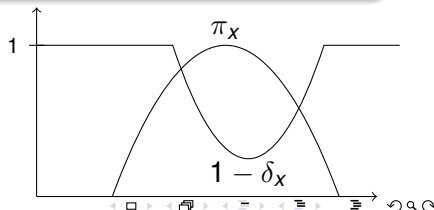
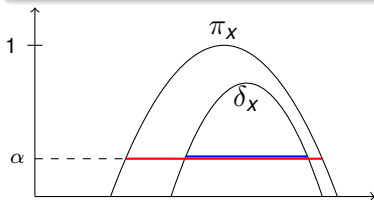
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Clouds : Introduction

Definition

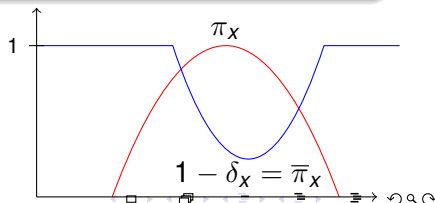
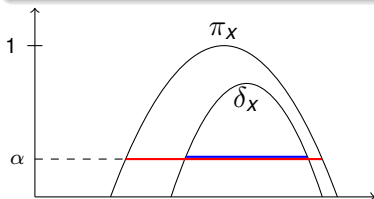
- A cloud can be viewed as a pair of dist. $[\delta(x) \leq \pi(x)]$
- A r.v. $X \in \text{cloud}$ iff $P(\{x|\delta(x) \geq \alpha\}) \leq 1 - \alpha \leq P(\{x|\pi(x) > \alpha\})$
- $P \in \mathcal{P}_\pi$ iff $1 - P(\pi(x) > \alpha) \geq \alpha$ (Dubois et al., 2004)
- And we have $1 - P(1 - \delta(x) > \beta) \geq \beta$ with $\beta = 1 - \alpha$
- $1 - \delta = \bar{\pi}$ and π are possibility distributions
-



Clouds : Introduction

Definition

- A cloud can be viewed as a pair of dist. $[\delta(x) \leq \pi(x)]$
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- And we have $1 - P(1 - \delta(x) > \beta) \geq \beta$ with $\beta = 1 - \alpha$
- $1 - \delta = \bar{\pi}$ and π are possibility distributions
- We have that $\mathcal{P}_{\text{cloud}} = \mathcal{P}_\pi \cap \mathcal{P}_{1-\delta=\bar{\pi}}$ (Dubois & Prade 2005)



Discrete clouds : formalism

Discrete clouds as collection of sets

Discrete clouds can be viewed as two set collections

- $\emptyset = A_0 \subset A_1 \subseteq A_2 \subseteq \dots \subseteq A_n \subset A_{n+1} = X \quad (\pi_X)$
- $\emptyset = B_0 \subset B_1 \subseteq B_2 \subseteq \dots \subseteq B_n \subset B_{n+1} = X \quad (\delta_X)$
- $B_i \subseteq A_i \quad (\delta_X \leq \pi_X)$

with constraints

- $P(B_i) \leq 1 - \alpha_i \leq P(A_i)$
- $1 = \alpha_0 > \alpha_1 > \alpha_2 > \dots > \alpha_n > \alpha_{n+1} = 0$

Relationship between clouds and generalized p-boxes

Theorem

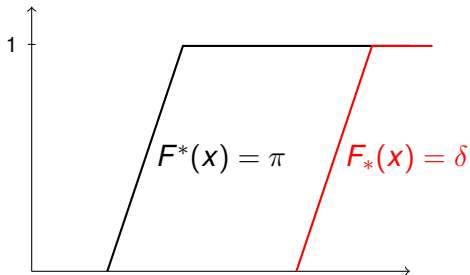
A generalized p-box is a particular case of cloud

Proof.

- $F^*(x) > F_*(x)$
- $F^*(x) \rightarrow$ possibility distribution π
- $F_*(x) \rightarrow$ distribution δ and $\bar{\pi} = 1 - \delta$ is a possibility distribution
- Gen. P-box equivalent to the cloud $[\delta, \pi]$



Illustration



Relationship between clouds and generalized p-boxes

Theorem

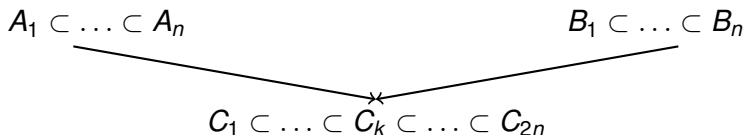
A cloud is a gen. p-box iff the set $\{A_i, B_j\} \ i, j = \{1, \dots, n\}$ forms a complete order with respect to inclusion

$(\forall i, j \ A_i \subseteq B_j \text{ or } A_i \supseteq B_j)$

Proof.

Idea : mapping constraints defining a discrete cloud into constraints defining a generalized p-box. □

Relationship between clouds and generalized p-boxes



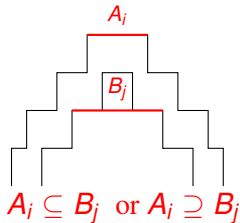
$$P(B_i) \leq 1 - \alpha_i \leq P(A_i) \longrightarrow \gamma_k \leq P(C_k) \leq \beta_k$$

$$\longrightarrow \text{if } C_k = A_i, \gamma_k = 1 - \alpha_i, \beta_k = \min\{1 - \alpha_j : A_j \subseteq B_i\}$$

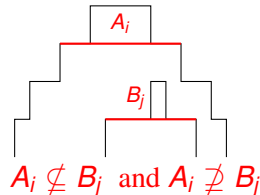
$$\longrightarrow \text{if } C_k = B_i, \beta_k = 1 - \alpha_i, \gamma_k = \max\{1 - \alpha_j : A_j \subseteq B_i\}$$

Corollary

A cloud $[\delta, \pi]$ is a generalized p-box iff δ, π are comonotonic



Comonotonic cloud



Non-comonotonic cloud

Characterization of non-comonotonic clouds

Theorem

Lower probability induced by a non-comonotonic cloud is not 2-monotone: $\exists A, B \subset X$ s.t. $\underline{P}(A \cap B) + \underline{P}(A \cup B) < \underline{P}(A) + \underline{P}(B)$

Proof.

(Chateauneuf, 1994) m_1, m_2 2 rand. sets. with focal sets $\mathcal{F}_1, \mathcal{F}_2$ and \mathcal{Q} the set of normalized joint rand. sets. s.t. if $Q \in \mathcal{Q}$

- $Q(A, B) > 0 \Rightarrow A \times B \in \mathcal{F}_1 \times \mathcal{F}_2$
- $A \cap B = \emptyset \Rightarrow Q(A, B) = 0$
- $m_1(A) = \sum_{B \in \mathcal{F}_2} Q(A, B)$ and $m_2(B) = \sum_{A \in \mathcal{F}_1} Q(A, B)$

Finding $\underline{P}(E)$ on $\mathcal{P}_{Bel_1} \cap \mathcal{P}_{Bel_2}$ is equivalent to finding

$$\min_{Q \in \mathcal{Q}} \sum_{(A \cap B) \subseteq E} Q(A, B)$$

→ using this result on $\mathcal{P}_{cloud} = \mathcal{P}_\pi \cap \mathcal{P}_{1-\delta}$



Non-comonotonic clouds : 4 sets case

Let us consider the following cloud

sets A_1, A_2, B_1, B_2

$A_1 \subset A_2, B_1 \subset B_2$

$B_i \subset A_i$

and constraints

$$P(B_1) \leq 1 - \alpha_1 \leq P(A_1)$$

$$P(B_2) \leq 1 - \alpha_2 \leq P(A_2)$$

$$1 > \alpha_1 > \alpha_2 > 0$$

with added constraint $A_1 \cap B_2 \neq \{A_1, B_2, \emptyset\}$ (non-comonotonicity).

$\pi, \bar{\pi} = 1 - \delta$ are respectively equivalent to belief functions

π	$\bar{\pi} = 1 - \delta$
$m(A_1) = 1 - \alpha_1$	$m(B_0^c = X) = 1 - \alpha_1$
$m(A_2) = \alpha_1 - \alpha_2$	$m(B_1^c) = \alpha_1 - \alpha_2$
$m(A_3 = X) = \alpha_2$	$m(B_2^c) = \alpha_2$

	$B_0^c = X$	B_1^c	B_2^c	
A_1	$A_1 \cap B_0^c \neq \emptyset$	$A_1 \cap B_1^c \neq \emptyset$	$A_1 \cap B_2^c \neq \emptyset$	$1 - \alpha_1$
A_2	$A_2 \cap B_0^c \neq \emptyset$	$A_2 \cap B_1^c \neq \emptyset$	$A_2 \cap B_2^c \neq \emptyset$	$\alpha_1 - \alpha_2$
$A_3 = X$	$A_3 \cap B_0^c \neq \emptyset$	$A_3 \cap B_1^c \neq \emptyset$	$A_3 \cap B_2^c \neq \emptyset$	α_2
	$1 - \alpha_1$	$\alpha_1 - \alpha_2$	α_2	

$$\underline{P}(A_1) = 1 - \alpha_1, \underline{P}(B_2^c) = \alpha_2$$

	$B_0^c = X$	B_1^c	B_2^c	
A_1	$A_1 \cap B_0^c \neq \emptyset$ $1 - \alpha_1$	$A_1 \cap B_1^c \neq \emptyset$ 0	$A_1 \cap B_2^c \neq \emptyset$ 0	$1 - \alpha_1$
A_2	$A_2 \cap B_0^c \neq \emptyset$ 0	$A_2 \cap B_1^c \neq \emptyset$ $\alpha_1 - \alpha_2$	$A_2 \cap B_2^c \neq \emptyset$ 0	$\alpha_1 - \alpha_2$
$A_3 = X$	$A_3 \cap B_0^c \neq \emptyset$ 0	$A_3 \cap B_1^c \neq \emptyset$ 0	$A_3 \cap B_2^c \neq \emptyset$ α_2	α_2
	$1 - \alpha_1$	$\alpha_1 - \alpha_2$	α_2	

$$\underline{P}(A_1) = 1 - \alpha_1, \underline{P}(B_2^c) = \alpha_2, \underline{P}(A_1 \cap B_2^c) = 0$$

	$B_0^c = X$	B_1^c	B_2^c	
A_1	$A_1 \cap B_0^c \neq \emptyset$ $1 - \alpha_1 -$ $\min(1 - \alpha_1, \alpha_2)$	$A_1 \cap B_1^c \neq \emptyset$ 0	$A_1 \cap B_2^c \neq \emptyset$ $\min(\alpha_2, 1 - \alpha_1)$	$1 - \alpha_1$
A_2	$A_2 \cap B_0^c \neq \emptyset$ 0	$A_2 \cap B_1^c \neq \emptyset$ $\alpha_1 - \alpha_2$	$A_2 \cap B_2^c \neq \emptyset$ 0	$\alpha_1 - \alpha_2$
$A_3 = X$	$A_3 \cap B_0^c \neq \emptyset$ $\min(1 - \alpha_1, \alpha_2)$	$A_3 \cap B_1^c \neq \emptyset$ 0	$A_3 \cap B_2^c \neq \emptyset$ $\alpha_2 - \min(1 - \alpha_1, \alpha_2)$	α_2
	$1 - \alpha_1$	$\alpha_1 - \alpha_2$	α_2	

$$\underline{P}(A_1) = 1 - \alpha_1, \underline{P}(B_2^c) = \alpha_2, \underline{P}(A_1 \cap B_2^c) = 0$$

$$\underline{P}(A_1 \cup B_2^c) = \alpha_2 + 1 - \alpha_1 - \min(\alpha_2, 1 - \alpha_1) = \max(\alpha_2, 1 - \alpha_1)$$

	$B_0^c = X$	B_1^c	B_2^c	
A_1	$A_1 \cap B_0^c \neq \emptyset$	$A_1 \cap B_1^c \neq \emptyset$	$A_1 \cap B_2^c \neq \emptyset$	$1 - \alpha_1$
A_2	$A_2 \cap B_0^c \neq \emptyset$	$A_2 \cap B_1^c \neq \emptyset$	$A_2 \cap B_2^c \neq \emptyset$	$\alpha_1 - \alpha_2$
$A_3 = X$	$A_3 \cap B_0^c \neq \emptyset$	$A_3 \cap B_1^c \neq \emptyset$	$A_3 \cap B_2^c \neq \emptyset$	α_2
	$1 - \alpha_1$	$\alpha_1 - \alpha_2$	α_2	

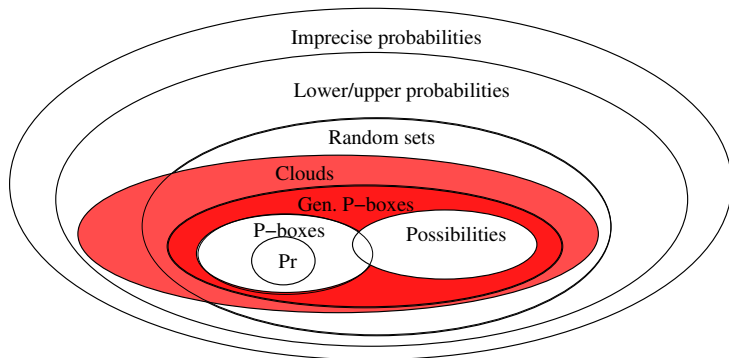
$$\begin{aligned} \underline{P}(A_1) &= 1 - \alpha_1, \underline{P}(B_2^c) = \alpha_2, \underline{P}(A_1 \cap B_2^c) = 0 \\ \underline{P}(A_1 \cup B_2^c) &= \max(\alpha_2, 1 - \alpha_1) \\ &\Rightarrow \max(\alpha_2, 1 - \alpha_1) < 1 - \alpha_1 + \alpha_2 \end{aligned}$$

	$B_0^c = X$	B_1^c	...	B_{i-1}^c	.	B_j^c	...	B_n^c
A_1	q_{11}	q_{12}	...	q_{1i}	.	$q_{1(j+1)}$...	$q_{1(n+1)}$
\vdots	\vdots	\vdots	\ddots	\vdots	.	\vdots	\ddots	\vdots
A_i	q_{i1}	q_{i2}	...	q_{ii}	.	$q_{i(j+1)}$...	$q_{i(n+1)}$
\vdots	\vdots	\vdots	\vdots	\vdots	.	\vdots	\ddots	\vdots
A_{j+1}	$q_{(j+1)1}$	$q_{(j+1)2}$...	$q_{(j+1)i}$.	$q_{(j+1)(j+1)}$...	$q_{(j+1)(n+1)}$
\vdots	\vdots	\vdots	\vdots	\vdots	.	\vdots	\ddots	\vdots
A_n	q_{n1}	q_{n2}	...	q_{ni}	.	$q_{n(j+1)}$...	$q_{n(n+1)}$
$A_{n+1} = X$	$q_{(n+1)1}$	$q_{(n+1)2}$...	$q_{(n+1)i}$.	$q_{(n+1)(j+1)}$...	$q_{(n+1)(n+1)}$

General case

From any non-comonotonic cloud, we can extract a 2×2 matrix $(A_i \cap B_j \neq \{A_i, B_j, \emptyset\})$ and make a similar reasoning.

Relations : graphical summary



Approximation of non-comonotonic clouds

Since capacities that are not 2-monotone can be difficult to handle, it is desirable to find approximations easy to compute and handle

Theorem

The following bounds provide an outer approximation of $[P_(A), P^*(A)]$ of $P(A)$ where $P \in \mathcal{P}_{\delta, \pi}$:*

$$\max(N_{\pi}(A), N_{1-\delta}(A)) \leq P(A) \leq \min(\Pi_{\pi}(A), \Pi_{1-\delta}(A)) \quad \forall A \subset X$$

Proof.

Immediate, since we know that $\mathcal{P}_{cloud} = \mathcal{P}_{\pi} \cap \mathcal{P}_{1-\delta}$ □

Approximation of non-comonotonic clouds

Theorem

Given the sets $\{B_i, A_i, i = 1, \dots, n\}$ forming the distributions (δ, π) of a cloud and the corresponding α_i , the belief and plausibility measures of the random set s.t.

$m(A_i \setminus B_{i-1}) = \alpha_{i-1} - \alpha_i$ are inner approximations of $\mathcal{P}_{\delta, \pi}$.

Proof.

$A_i \setminus B_{i-1} \neq \emptyset$ (by definition) and the random set given above is always coherent with the marginal random sets induced by δ, π . Bounds are exact in case of comonotonicity. \square

Results summary

- A gen. P-box is a special case of random set and can be represented by two possibility distributions
- Clouds are also equivalent to a pair of possibility distributions.
 - Comonotonic clouds are equivalent to generalized p-boxes and thus are a special case of random sets
 - Non-comonotonic clouds are not 2-monotone in the general case, but they can be easily approximated.

Perspectives, Open problems and questions

Continuous case on the real line

If our propositions hold in the continuous case on the real line, then a comonotonic cloud can be characterized by a continuous belief function (Similar to Smets, 2005) with uniform mass density, whose focal elements would be disjoint sets of the form $[x(\alpha), u(\alpha)] \cup [v(\alpha), y(\alpha)]$ where $\{x : \pi(x) \geq \alpha\} = [x(\alpha), y(\alpha)]$ and $\{x : \delta(x) \geq \alpha\} = [u(\alpha), v(\alpha)]$.

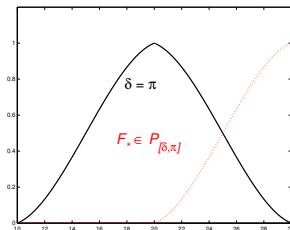
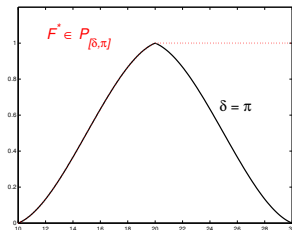
This remains to be proved.

Perspectives, Open problems and questions

The special case of *thin* clouds

A *thin* cloud ($\delta = \pi$) contains an ∞ of probability distributions (Dub. & Pra. 2005). The induced continuous bel. f. would be a uniform mass density distributed over doubletons $\{x(\alpha), y(\alpha)\}$

Cumulative distributions F^*, F_* are in the *thin* cloud



Perspectives, Open problems and questions

Towards generalization

Are results of the type

- $\mathcal{P}_{cloud} = \mathcal{P}_{\pi} \cap \mathcal{P}_{1-\delta}$
- Structure of comonotonic clouds and gen. p-boxes are similar, and are ∞ -monotone capacities
- Non-comonotonic clouds are not 2-monotone

still true in spaces X of infinite cardinality ? and under which (topological, measurability, ...) assumptions ?

Perspectives, Open problems and questions

Operations on Gen. p-boxes and clouds

Are operations of

- Fusion
- Conditioning
- Propagation through a mathematical model

easy to define for gen. p-boxes and clouds ? which property of the representations do they conserve ? (e.g.

$[\delta_1, \pi_1] \cap [\delta_2, \pi_2] = [\max(\delta_1, \delta_2), \min(\pi_1, \pi_2)]$ is easy to compute and is still a cloud, but can be expected to be an inner approx.

of $\mathcal{P}_{[\delta_1, \pi_1]} \cap \mathcal{P}_{[\delta_2, \pi_2]}$)